THE

MECHANICS OF ENGINEERING

INTENDED FOR USE IN UNIVERSITIES,

AND IN COLLEGES OF ENGINEERS.

BY

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CAMBRIDGE:

PRINTED AT THE UNIVERSITY PRESS:

FOR JOHN W. PARKER, WEST STRAND, LONDON;

AND J. & J. DEIGHTON, CAMBRIDGE.

M.DCCC.XLI.

TO

THE REV. ROBERT WILLIS,

JACKSONIAN PROFESSOR IN THE UNIVERSITY
OF CAMBRIDGE.

MY DEAR WILLIS,

IT is but just that I should dedicate this work to you. Without your exhortations I should not have written it; and I have in various parts received from you suggestions and information which have given it a great part of its value, whatever that may amount to. The separation of the two sciences of Mechanics and Mechanism, which we have long seen to be indispensable to the practical application of theoretical Mechanics, and which I have here kept in view, will, I hope, soon receive its full completion by the publication of your *Treatise on Mechanism;* since our proof sheets have been going through the press at the same time, and yours are drawing on to the termination which mine have reached. I think with admiration of the combination of practical knowledge

and theoretical beauty which I know to belong to your work, and which it is so rare an endowment to be able to give: and I anticipate with pleasure this addition to the numerous benefits which you have conferred upon the world of art and science, by your inventions and speculations.

Believe me, my dear Willis,

Most sincerely yours,

W. WHEWELL.

TRINITY COLLEGE,
May 1, 1841.

PREFACE.

I WOULD willingly have avoided writing the present Work, and should not have done so, if the subject had not appeared to be, to a certain extent, put into my hands by the circulation of my former works on Mechanics. Various circumstances at the present time make it desirable that the subject of Engineering should be treated in such a mode that it may be made a satisfactory part of a liberal education. I refer not only to the attempts now so laudably making in various quarters to improve the professional education of Engineers, but also to the desire which is more and more felt in the country, that what our students learn of mathematics in their university career should have some meaning in real life. In the science of Mechanics it has especially happened that the mathematical study of the subject has been pursued with very little regard to its practical application. The consequence of this is, not only that our theoretical teaching is of little value in preparing a person for any part of the business of Engineering, but also, that it is of little value as intellectual discipline. For the student has not been taught to seek and to find

in the mechanism which he sees about him the exemplification of his theoretical principles; and hence he never learns to think steadily upon the subject, and when his days of pupilage are past, ceases to think upon it at all. This could hardly happen if his education made him familiar with principles readily applicable to every machine and every structure which came in his way: for in that case he would be constantly stimulated to understand what he saw; and clear views of mechanical relations would become part of the habits of his mind. The relations of space once learnt in geometry do not fade away from our thoughts, because throughout our lives we continue familiar with exemplifications of them in Geography, Astronomy, and other common pursuits. If the common Problems of Engineering were to form part of our general teaching in Mechanics, this science also might become a permanent possession of liberally educated minds. Every roof, frame, bridge, oblique arch, machine, steam-engine, locomotive carriage, might be looked upon as a case to which every well-educated man ought to be able to apply definite and certain principles in order to judge of its structure and working. And this would, I conceive, be an improvement, not only in professional, but in general education.

The present Work is in some degree constructed with the view of answering this purpose. I have introduced into it the subjects I have mentioned. In speaking of Machines, I have rejected the usual classification of the

"Mechanical Powers;"—a tradition derived from the later Greek writers, and quite unsuited to the present state of Mechanism. The classification of the Elements of Machines which I have adopted, (in Chapter I.) is borrowed in a great measure from that which Professor Willis has usually employed in his Lectures, and is about to publish in his work on Mechanism. The subject of "Oblique Arches" has not, so far as I have seen, been hitherto treated on Statical principles. As one of the most remarkable steps ever made in structure, it cannot fail to excite the curiosity of the intelligent student of Mechanics. The results to which I am led (Chap. IV. Sect. 2, and Chap. V. Sect. 4,) are of some elegance. It may perhaps be thought that the solution of the problem in which we reject friction, is a mere useless exercise of ingenuity; but I think the student will find it aid him much in forming his conception of the relations of the forces by which Oblique Arches are supported. The Chapters upon Labouring Force are in a considerable degree borrowed from French writers, (Navier, Poncelet, and others,) who have lately prosecuted this subject with great zeal, and in a very instructive manner. I have attempted, however, to present this matter in a more systematic shape than my predecessors. I have ventured to employ the term "Labouring Force," for the French "Travail." I also found it absolutely necessary to employ a constant term for the "dynamical unit," *one pound raised one foot;* for it is impossible to proceed in calculations on this subject, without being able to describe in one single numerical expression the various

forms of the same thing;—one pound raised twelve feet; two pounds raised six feet; three pounds raised four feet; six pounds raised two feet; and so on. I have called this 12 *dynamical units*, and I should think that no inconvenience would arise from abbreviating this into "12 *dynams*."

The Chapter on Impact is compiled by combining what I formerly borrowed from Professor Airy on that subject, with the reasonings and conclusions contained in the works of Navier and Poncelet.

I fear the Work may be found incomplete in many respects; but to avert some objections on this head, I must beg my Engineer readers to bear in mind that I have not ventured to call it a book on *Engineering*, but on the *Mechanics of Engineering*. The business of the Engineer is of a wide and various kind, and might perhaps best be put in a systematic form by dividing it into a number of departments, and making each the subject of a scientific treatise. Thus, in addition to the Mechanics of the subject, we might have the Geometry, or the Pure Mathematics of Engineering;—the Hydrostatics and Hydrodynamics of Engineering; (this is usually called Hydraulics);—the Thermotics and Atmology of Engineering; (so much of the doctrines of the relations of heat and vapour as is required in the profession);—the Geology and Mineralogy of Engineering;—the Architecture of Engineering;—and perhaps some other similar parts. By

such a combination of parts, Engineering might really grow up into a subject of scientific certainty, and of constantly increasing completeness.

I should rejoice if the present Work should be accepted by those able to judge of the matter, as a contribution of any value towards such a system.

TRINITY COLLEGE,
May 1, 1841.

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PRINCIPLES OF MECHANICS.

In the present volume some of the principles of Mechanics are traced to their exemplifications in the practical business of the Engineer.

The principal subjects here treated of are *Machines*, *Structures*, *Strength of Materials*, and *Labouring Force*.

Combinations of material parts, when put together in order to move and produce certain motions, and thus to do *work*, are called *Machines;* when constructed with a view to support weights, or to resist forces, without being moved, they are termed *Structures*.

The conditions of equilibrium, and of force exerted in motion, which are exemplified in machines, belong to the science of Mechanics; and may be deduced from the principles of Mechanics elsewhere stated. But if we consider the effect of machines in transmitting and modifying the velocity and direction of motion, the investigation falls under another science, which is termed *Pure Mechanism*, or *Kinematics*.

When by means of any machine work is done, labouring force is *expended* or *consumed;* and the measure of this force, and the laws which regulate its expenditure and consumption, belong also to the subject of the present volume.

CHAPTER I.

MACHINES.

1. MACHINES are assemblages of material parts, or *Pieces*, constructed for the purpose of transmitting either force or motion from one part to another, and of modifying in various ways the motion and the force. We consider here the latter object.

2. *Simple* machines are those which consist of one or two pieces. Those which consist of more are *compound* machines. The pieces in a machine affect and determine each other's motions; and the motion of a piece may also be *constrained* in the following ways.

(1.) By having a *fixt axis* or center, about which it is moveable. Thus a *lever* is a linear piece moveable about an axis in it; and the axis is the *fulcrum*. Any plane piece moveable about an axis perpendicular to its plane, may be termed a *wheel*.

(2.) By *sliding*; as when a body has a plane surface, and slides in contact with a plane; or when it has a projection which slides in a groove.

These two modes of constraint are in fact identical. Thus a wheel may turn by having two projecting pins which slide in a circular groove; or by sliding round in a circular aperture which it fits. And if we suppose the fixt axis to be at an infinite distance, the circular motion of a piece connected with that axis becomes rectilinear sliding.

3. When motions are thus constrained, there are certain *constraining forces* excited, which produce this effect, and which balance or destroy all the forces by which the pieces tend to transgress this constraint.

4. Pieces in a machine act upon each other in the following ways.

(1.) By *attachment* at a point; as when a cord is fastened to a given point of a line; or a rod is fastened by a pin to a wheel; or a rod by a hook to a crank.

(2.) By *contact of surfaces;* as when a wedge presses an obstacle; or when a cam lifts a handle; or when one toothed wheel turns another.

(3.) By *contact of wrapping bands;* as when a cord constrains the motion of a pully round which it passes.

5. In the present Chapter we consider all sliding and turning to take place with perfect freedom, friction being neglected.

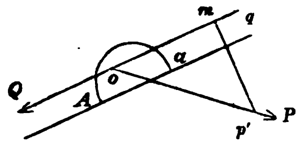
6. Hence the *line of mutual action* of two surfaces in contact will be perpendicular to the surfaces (Mech.* Int. 34).

7. Hence also when a wrapping band passes round a piece, the *tensions* of the two parts on the two sides of the piece *are equal* (Mech. Int. 27).

8. Prop. *If a piece (in a machine) constrained by rectilinear sliding without friction to move in a given direction, be urged by a force acting in that direction, and kept in equilibrium by another force; the first force will be equal to the second, multiplied by the cosine of the angle which*

* The references thus made refer to the Articles in the sixth edition of *An Elementary Treatise on Mechanics.*

the direction of the second force makes with the direction of constrained motion.



Let a piece A be constrained by rectilinear sliding without friction to move in a direction Aa; and let it be acted upon by a force Q in that direction, and a force P in the direction oP. Let Po, Qo meet in the point o, and let Qo be produced to m: then $Q = P.\cos Pom$.

For the forces P, Q, may each be supposed to act at any point of its direction, and therefore may both be supposed to act at o: and hence R, the resultant of the constraining forces, by which the body is compelled to slide, must also act in a direction passing through the point o (Mech. 23). And this force will be perpendicular to the direction of sliding, since there is no friction (Mech. Int. 34). Hence, draw pm perpendicular to om, and the three forces P, R, Q are in the directions op, pm, mo. And they are forces acting at a point, and keeping it in equilibrium. Therefore they are as the three lines op, pm, mo (Mech. 17). Hence $P : Q :: op : om :: 1 : \cos pom$. Wherefore

$$Q = P \cos Pom.$$

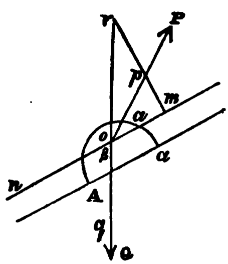
Cor. 1. In the same manner it appears that we have

$$R = P \sin Pom.$$

Cor. 2. The same is true when the piece slides in a curvilinear path, Aa being the tangent to this path.

9. Prop. *If a piece (in a machine), constrained by rectilinear sliding without friction to move in a given direction, be kept in equilibrium by two forces acting in any directions; these forces will be inversely as the cosines of the angles which their directions make with the direction of constrained motion.*

Let a piece A be constrained to move in the direction Aa, and be kept in equilibrium by two forces P, Q, acting in directions oP, oQ. Draw mon parallel to Aa, and we shall have $P : Q :: \cos Qon : \cos Pom$.



Let S be a force in the direction of constrained motion on which would balance P; therefore (Art. 8) $S = P \cos Pom$. Now S and Q are equivalent in their effect in the direction of constrained motion. But S acting in on would balance an equal force S acting in om. Therefore Q would balance a force S acting in om. Hence again (Art. 8) $S = Q \cos Qon$. Therefore, $Q \,.\, \cos Qon = P \,.\, \cos Pom$, and $P : Q :: \cos Qon : \cos Pom$.

Cor. 1. If mp perpendicular to om, meet oP in p, and Qo produced in r, the resultant R of the constraining forces must be in the direction mpr, and must pass through the point o (Mech. 23): and the three forces P, R, Q will be in the directions of the sides of the triangle opr, and therefore will be as these lines op, pr, ro.

Cor. 2. Hence $R = Q \sin Qon - P \sin Pom$.

For $pr = mr - mp = or \,.\, \sin \,.\, rom - op \,.\, \sin \,.\, pom$;

$$\text{but } \sin rom = \sin Qon;$$

whence by last Cor. $R = Q \sin Qon - P \sin Pom$,

Cor. 3. If P and Q be on the same side of mn, we shall find in like manner $R = Q \sin Qon + P \sin Pom$

10. Prop. *To determine the conditions of equilibrium of the Inclined Plane.*

The above proposition is applicable to the case of a heavy body supported upon an inclined plane by a force acting in any direction, without friction. In this case one of

the forces which acts upon the body is its own weight, which acts in a vertical direction. If Q be this force, the angle which the direction of constrained motion makes with the direction of Q, is the angle which the inclined plane makes with the vertical; let this angle be β. Also let α be the angle which P makes with the direction of constrained motion, that is, with the inclined plane. Hence by the proposition and its corollary,

$$P \cos \alpha = Q \cos \beta; \quad R = Q \sin \beta - P \sin \alpha.$$

Cor. 1. If P act along the plane,

$$\alpha = 0, \quad P = Q \cos \beta, \quad R = Q \sin \beta.$$

Cor. 2. If P act horizontally, α is the complement of β, and using Cor. 3 of Art. 9.

$$P \sin \beta = Q \cos \beta, \quad R = Q \sin \beta + P \cos \beta;$$

whence $P = Q \operatorname{cotan} \beta$; $R = \dfrac{Q}{\sin \beta}$.

Cor. 3. If P act vertically, α is equal to β:

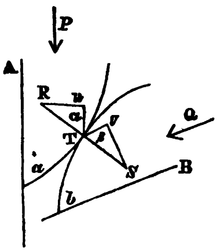
whence $P = Q$, $R = 0$.

11. Prop. *In a machine, if two pieces in one plane, acting on each other by contact, be constrained, by rectilinear sliding, to move in given directions, and be kept in equilibrium by two forces acting in those two directions; the two forces are as the cosines of the angles which the line of action (of one piece on the other,) makes with the two directions of constrained motion.*

Let A, B, be two pieces constrained to move in given directions Aa, Bb, and acted upon by forces P, Q in those directions. Let the two pieces touch in the point T, and let RTS be the line of action of one piece on the other. Let TU, TV be parallel to Aa, Bb, respectively; then

$$P : Q :: \cos RTU : \cos STV.$$

Let equal lines TR, TS represent the mutual action of each piece upon the other, and let RU be perpendicular upon TU, SV upon TV. The force exerted by the piece B on the piece A, which is represented by TR, may be resolved into two forces TU, UR; of which TU, being in the direction of the constrained motion of the piece A, is equal to P, the force which urges A in that direction; and the force UR, perpendicular to this, is destroyed by the constraining forces.



In like manner, the force exerted by the piece A on the piece B, which is represented by TS, may be resolved into two forces TV, VS; of which TV, being in the direction of the constrained motion of the piece B, is equal to Q, the force which constrains B in that direction, and the force VS is destroyed by the constraining forces. Hence

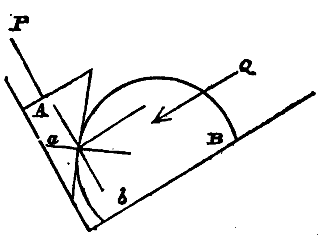
$$P : Q :: TU : TV :: TR \cos RTU : TR \cos STV;$$

$$\text{whence } P : Q :: \cos RTU : \cos STV.$$

Cor. If $RTU = \alpha$, $STV = \beta$, $P \cos \beta = Q \cos \alpha$.

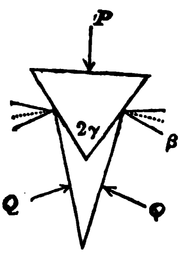
12. Prop. *To determine the condition of equilibrium of the Wedge.*

The above proposition may be applied to the action of a wedge. A wedge is a sliding piece (as A) which being urged in the sliding direction (Aa), by its surface exerts force upon a body (B) constrained to move in another direction (bB).



The wedge is often treated of as exerting force by

two surfaces upon two resisting obstacles, in two different directions in the same plane. In general, then, two surfaces or *sides* of the wedge, are conceived to be planes making equal angles with the direction in which the force P acts; and in this case, the constraint which produces sliding in this direction is the equality of the reaction exerted on the two sides of the wedge. The two plane sides of the wedge meet and form the *edge* of the wedge: and the wedge is supposed to be bounded also by another plane surface perpendicular to the direction of the force P, called the *back* of the wedge. The *angle of the wedge* is the angle which the sides form at the edge where they meet; the sliding direction bisects this angle.

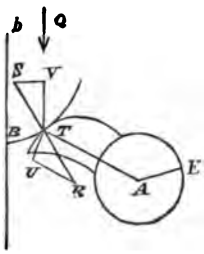


The line of action is perpendicular to the side of the wedge; and hence a is the complement of the angle which the side of the wedge makes with the sliding direction: that is, a is the complement of half the angle of the wedge: and if 2γ be the angle of the wedge, $\cos a = \sin \gamma$. Also β is the angle which the constrained motion of the obstacle makes with the perpendicular to the plane of contact. If the constrained motion be not rectilinear, the formula will still be applicable, taking the direction of the constrained motion of B at the first instant of the motion. Hence for each side of the wedge $P \cos \beta = Q \sin \gamma$; and $2P \cos \beta = 2Q \sin \gamma$: where $2P$ is the whole force which acts, and $2Q$ the whole resistance.

13. PROP. *If two pieces in a machine, acting upon each other by contact, be constrained in their motions, one by rectilinear sliding, and one by a fixt axis; and be urged, the one by a force acting in the direction of the rectilinear*

sliding, and the other by a force acting about the fixt axis; in equilibrium, the moment of this force is to the product of the other force by the radius at the point of contact, as the cosines of the angles which the line of action makes with the lines of constrained motion, respectively.

Let a piece, moveable about a fixt center A, act upon a piece B constrained to move parallel to Bb, and let T be the point of contact, RTS the line of action of the one piece on the other. Let equal lines TR, TS, represent the mutual action of the two pieces; let TA be the radius drawn from the center A to the point of contact T; let TU be perpendicular to TA; TV parallel to Bb; and let RU be perpendicular upon TU, SV upon TV; Q the force which urges B; $P \times AE$ the moment of the force about A.



The force exerted by the piece A on the piece B, which is represented by TS, may be resolved into two forces TV, VS; of which VS is destroyed by the constraining forces, and TV, being in the direction of the constrained motion of B, is equal to Q, the force which urges B in that direction. Also, the force exerted by the piece B on the piece A, which is represented by TR ($= TS$) may be resolved into TU, UR; of which UR is in the direction of the radius TA, and is destroyed by the reaction of the fixt axis; and TU, acting perpendicularly at the radius TA, has a moment equal to $P \times AE$, the moment of the forces which keep the piece A in equilibrium. (Mech. 7.)

$$\text{Hence } P \times AE = TU \times TA = TR \, . \cos RTU \times TA;$$

$$Q \times TA = TS \, . \cos STV \times TA.$$

$$\text{Whence } P \times AE : Q \times TA :: \cos RTU : \cos STV.$$

Also TU, being perpendicular to TA, is the line of constrained motion for the point T in the wheel TA.

COR. If $RTU = \alpha$, $STV = \beta$, we have

$$P \times AE . \cos \beta = Q \times TA . \cos \alpha.$$

14. PROP. *To determine the conditions of equilibrium of the Cam.*

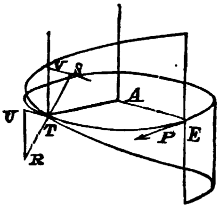
The above proposition is applicable to the action of a CAM, pressing a piston, or a stamper, or any other piece capable of sliding up and down. A *cam* is a curved piece, revolving in its own plane, and by its edge exercising pressure in that plane upon another piece.

In this case the force Q is the weight of the stamper, or the load upon the piston which acts in the direction of the sliding motion. The surface of the cam at the point of contact (friction being excluded) makes with the direction of sliding an angle of which the complement is β, and with the radius, an angle α. The moment $P \times AE$ is the moment of the force which tends to turn the wheel, and thus to raise the piston or stamper by means of the cam.

15. PROP. *To determine the condition of equilibrium of the Screw.*

In the above proposition, Art. 13, the motions and pressures are all supposed to take place in a plane perpendicular to the axis of motion. But if we suppose the line of sliding Bb not to be in the plane TA perpendicular to the axis at A, and if the two pieces touch each other at T in a surface oblique to TA, the equilibrium is still possible, the axis being fixt in the direction of its length; and the same proposition will apply.

Hence the proposition is applicable to a SCREW. A screw is a piece moveable round a fixt axis, and carrying a surface oblique to the axis, which acts by contact upon a piece which can slide parallel to the fixt axis.



In the first place, let the plane of contact at T be a plane passing through the radius TA. This plane of contact makes with the *transverse section* of the screw (the plane perpendicular to the fixt axis) an angle β, equal to the angle STV, which the line of action makes with the direction of sliding. For, friction being neglected, the line of action is perpendicular to the plane of contact; and the line of sliding is perpendicular to the transverse section of the screw.

Also the line of constrained motion of the point T about A is perpendicular to TA, and is in the transverse section; and the line of action is also perpendicular to TA, and makes with a perpendicular to the transverse section an angle TRU, whose complement is α: hence α is the complement of β.

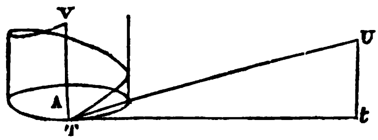
$$\text{Hence we have } P \times AE \times \cos\beta = Q \times TA \times \sin\beta.$$

$$\text{And hence } P \times AE = Q \times TA \times \tan\beta.$$

16. The surface oblique to the axis, in the usual form of the screw, is continued, cutting the surface of a cylinder which has the fixt axis for its axis, and cutting it in such a manner as always to make, at the cylindrical surface, the same angle with the transverse section: which angle is, above, denominated β. The oblique surface thus forms a spiral line upon the surface of the cylinder, which line is called the *thread* of the screw. If the surface of the cylinder, being slit along a line parallel to the axis, were unwrapt, so as to become a plane, each revolution of the thread round

the cylinder would become a straight line upon the unwrapt surface; and this straight line would make an angle β with the line perpendicular to the axis, which line, in the unwrapt surface, represents the transverse section.

Thus let the cylinder be slit up at the line TV parallel to the axis, and unwrapt: the transverse section becomes the straight line Tt, perpendicular to TV, and the thread becomes the straight line TU, making an angle β with Tt.



The line Tt will be equal to the circumference of the circle whose radius is AT, and TU will be the corresponding circumference of the thread.

At V another circumference of the thread begins, and TV or tU is the *distance of two successive circumferences of the thread.*

17. Prop. *In the screw, when* P *and* Q *are in equilibrium,* P *is to* Q *as the distance of two successive circumferences of the thread is to a whole circumference described by the extremity of the arm at which* P *acts.*

We have already seen (Art. 15) that in the screw

$$P \times AE = Q \times TA \times \tan\beta.$$

Hence 2π being the circumference of a circle of radius 1,

$$P \times 2\pi \times AE = Q \times 2\pi \times TA \times \tan\beta.$$

But $2\pi \times TA$ is the circumference of the circle whose radius is TA, and is therefore Tt: and $2\pi \times AE$ is the whole circumference described by the point E: hence

$$P \times \text{circumference described by } E = Q \times Tt \times \tan\beta.$$

But $Tt \times \tan \beta = tU =$ dist. of succeeding threads.

Hence $P \times$ circumf. desc. by $E = Q \times$ dist. of succ. threads.

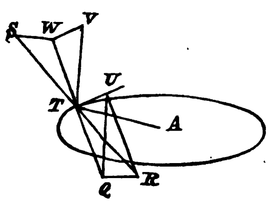
And $P : Q ::$ dist. of succ. threads : circumf. desc. by E.

18. The thread of the screw in practice is not a mathematical line, but a strip of material projecting from the cylinder; and the spiral projection thus produced may be called the ***solid thread*** of the screw.

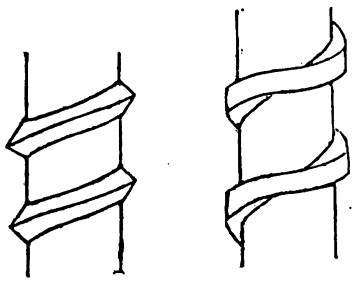
The sliding piece is usually constructed with a cavity fitted to contain the whole or a portion of this solid thread, so that the screw and the sliding piece are in contact, not at a point only, but for a considerable portion of surface.

If the external piece be fixt, and the screw be turned round its axis, the axis will exert force in the direction of its length, and the same propositions will apply.

19. According to the supposition made above, that the plane of contact passes along the radius AT, the strip of surface which bounds the solid thread will be at every point perpendicular to the surface of the cylinder round which it winds. But now let the plane of contact not cut the transverse section in the line TA. In this case, the line of action STR is no longer in the plane perpendicular to TA. Let RQ be let fall perpendicular upon the plane TUQ, which is perpendicular to TA; and let QU be drawn perpendicular to TU, which is the direction of T's motion. Then the equal lines TR, TS representing the mutual action of the screw and its thread, the force TR is equivalent to TQ, QR, of which QR is destroyed by the transverse reaction of the fixt axis. And TQ is equivalent to TU, UQ; of which UQ is destroyed by the longitudinal reaction of the fixt axis.



In like manner, SW being drawn perpendicular to the plane QTU, and WV parallel to TU, the force TS is equivalent to TV, VW, WS, of which VW, WS are destroyed by the constraining forces which compel the load to slide parallel to the axis. And since SW, RQ are perpendicular to the plane QTU, they lie in one plane, and QW, the intersection of the two planes, is a straight line: hence the triangles STW, RTQ are equiangular; and since $RT = TS$, they are equal; whence $QT = TW$. Hence the same reasoning will apply in this case as in Article 15.



Hence the propositions concerning the screw are not altered by any differences in the form of the thread, such as those represented in the margin.

20. *In the* ENDLESS SCREW *the piece acted upon by the screw is not a sliding piece, but a wheel revolving about a fixt axis, the axis of the screw being in the plane of the wheel.*

If B be the center of the wheel, and $Q'.BF$ the moment of the force which turns it, it is easily seen that in the case of equilibrium we must have

$$Q'.BF = Q.BT,$$

Q being the same as in Art. 15.

$$\text{And, by Art. 15,} \quad P.AE = Q.AT.\tan\beta.$$

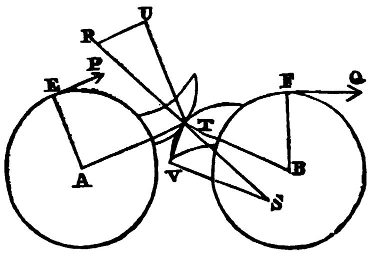
$$\text{Hence} \quad P.AE.BT = Q'.BF.AT\tan\beta,$$

$$\text{whence} \quad P : Q' :: BF.AT.\tan\beta : AE.BT.$$

21. PROP. *If two pieces in a machine, acting upon each other by contact, be moveable about parallel fixed*

axes in one plane, and be urged each by forces acting about its axis; in the case of equilibrium, the moments of the forces about the two axes are directly as the radii at the point of contact, multiplied by the cosines of the angles which the line of action makes with the lines of constrained motion respectively.

Let a piece moveable about a fixed center A act upon a piece moveable about a fixed center B, and let T be the



point of contact, RTS the line of action of the one piece on the other.

Let TA, TB be the radii drawn from the point of contact T to the centers A and B; let equal lines TR, TS, represent the mutual action of the two pieces: let TU be perpendicular to TA, and TV perpendicular to TB in the plane of motion; and let RU be perpendicular to TU, SV to TV; also let $P \times AE$, $Q \times BF$, be the moments of the forces which act about the centers A and B.

The force exerted by the piece A on the piece B, which is represented by TS, may be resolved into two forces TV, VS, of which VS is destroyed by the reaction of the center B, and TV acting perpendicularly at the radius TB, has its moment equal to $Q \times BF$, the moment of the force which keeps the piece B in equilibrium. In like manner TU, the effective part of the force which B exerts on A, has

its moment equal to $P \times AE$, the moment of the force which keeps the piece A in equilibrium. Hence

$$P \times AE = TU \times TA = TR \cos RTU \times TA,$$
$$Q \times BF = TV \times TB = TS \cos STV \times TB.$$

Hence

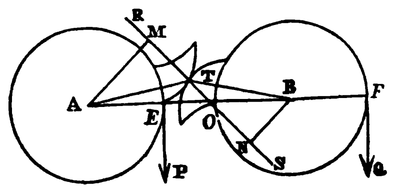
$$P \times AE : Q \times BF :: TA \cos RTU : TB \cos STV.$$

Cor. 1. Since ATU, BTV are right angles, we have

$$P \times AE : Q \times BF :: TA \sin ATR : TB \sin BTS.$$

Cor. 2. If we draw AM, BN perpendiculars upon RTS,

$$AM = TA \sin ATR,$$
$$BN = TB \sin TBS:$$



hence, mom. about A : mom. about B :: AM : BN.

Cor. 3. Let the line of action cut the line of centers AB, as at O; by similar triangles AOM, BON,

$$AM : BN :: AO : BO.$$

Hence mom. about A: mom. about B :: AO : BO.

22. The above proposition and its corollaries are applicable to toothed wheels. In general, toothed wheels have several teeth occupying the whole circumference, so that one wheel can turn the other continually.

The object of toothed wheels being to communicate motion, various problems arise concerning them: as, that the motion of one wheel being uniform, that of the other shall also be uniform; and the like. But these problems belong to Mechanism, not to Mechanics proper. See Willis, *Principles of Mechanism*, Chap. III.

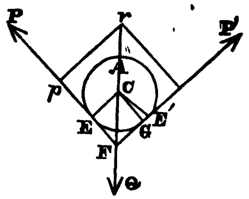
23. In the above proposition, the axes of the wheels are supposed to be parallel and the pressures to be in the same

plane. But if we suppose two wheels to be moveable about fixt axes in any situations whatever, and to act upon each other by a surface of contact in any direction, similar reasoning will hold, and therefore Cor. 2 of the proposition will still be true, AM, BN being lines perpendicular both to the line of action, and to the axes A, B respectively.

Hence the proposition is applicable to the case of wheels with axes not parallel, turning each other. Such cases are known in practice by the designations of crown-wheels, face-geer, bevil wheels, and others.

24. Prop. *If a piece in a machine be constrained in its motion by a wrapping band, sliding without friction, and kept in equilibrium by a force acting upon it; this force will act in a direction bisecting the angle made by the two extremities of the band, and will be equal to the cosine of the half of this angle, multiplied into double the tension of the band.*

Let A be the piece, constrained in its motion by the wrapping band $PEE'P'$, and acted upon by the force Q. In the condition of equilibrium, the band must necessarily be pulled by equal forces at its two ends, since there is no intermediate force which acts in the course of its length $PEE'P'$ (friction being excluded). (Mech. Int. 27.)



Hence the piece is kept in equilibrium by three forces; namely, the two equal forces P, P' acting in the directions EP, $E'P'$ and the force Q, acting in the direction CQ.

Also the force exerted by the band upon the piece A will be the same as if it were fastened to the piece at E, E'; for in the condition of equilibrium, the band has no tendency to slide along the piece, and therefore may be supposed to be fastened. (Mech. Int. 36.)

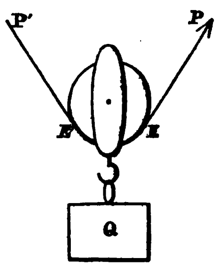
Let the directions of the two forces, PE, $P'E'$ meet in F; the direction of the third force must also pass through F, for otherwise there could not be equilibrium. Let EC, perpendicular to EF, meet QF in C, and draw CG perpendicular to FE'. The force Q may be supposed to act at C, so as to keep C immoveable. But this is equivalent to supposing C the fulcrum of a lever ECG, on which the forces P, P' balance. And since the forces P, P' are equal, and balance about C, the arms, CE, CG must be equal. Hence the right-angled triangles CFE, CFG, which have a common side CF, are equal in all respects, and the angles CFE, CFG are equal. Therefore the direction CQ in which the force Q acts, bisects the angle PFP', made by the two extremities of the band.

Also the three forces P, P', Q may be supposed to act at the same point F, and to keep each other in equilibrium: and hence Q will be equal to the resultant of the other two P, P'. And hence (Mech. 16.) $Q = 2P \cos CFE$.

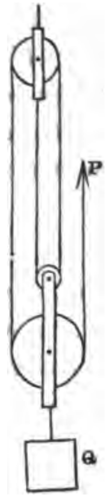
Cor. If $PFP' = 2a$, $Q = 2P\cos a$.

25. Prop. *To determine the condition of equilibrium in the single moveable pully.*

The above proposition applies to a pully. A pully consists of a *block* and *sheave*: the sheave is a small wheel moveable about an axis fixt in the block, along the edge of which wheel the string runs. The moveable sheave enables the string to run with very little friction; and hence the preceding proposition is approximately applicable. The force Q acts upon the block, and P is the tension of the string: also $2a$ is the angle made by the two parts PE, $E'P$ of the string.



Cor. 1. If the strings are parallel, $Q = 2P$.

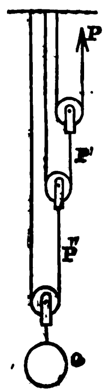


COR. 2. The same string may pass several times through each block, and if there be n strings at the block at which Q acts, and the strings be parallel, $Q = nP$.

For the tension of each of the strings at that block will be P, and since they are parallel, P is equal to the sum of them.

The essential pieces in the pully, as a mechanical instrument, are two, the string and the block; and hence it is a simple machine.

26. By the combination of several of the above simple machines are formed compound machines, consisting of three or more pieces; the force which corresponds to P in one simple machine, being made to correspond to Q in another connected with it. Hence the relation between the extreme forces in such combinations is found by eliminating the intermediate forces. This will appear in the following examples.



27. PROP. *In a system of pullies in which each pully hangs by a separate string, and the strings are parallel; if* n *be the number of moveable pullies,* $Q = 2^n P$.

The tension of the first string is P; let the tension of the second be P', of the third P'', and so on. Hence we have

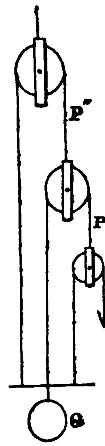
$$Q = 2P'', \quad P'' = 2P', \quad P' = 2P;$$

and there is one equation for each pully.

$$\text{Hence } Q = 2^n P.$$

28. PROP. *In a system of pullies in which each string is attached to the piece at which* Q *acts, and the strings are parallel, if* n *be the number of moveable pullies,* $Q = (2^n - 1) P$.

The tension of the first string is P, let that of the second be P', of the third P''; then $P' = 2P$, $P'' = 2P' = 2^2P$, and so on. And all the strings being attached to the piece on which Q acts, and being parallel, Q must be equal to the sum of them. Hence



$$Q = P + P' + P'',$$
$$= P + 2P + 2^2P.$$

This is a geometrical series with the ratio 2; and the number of terms will be n, the number of strings. Also the first term is P, and the last is $2^{n-1}P$: hence the sum is

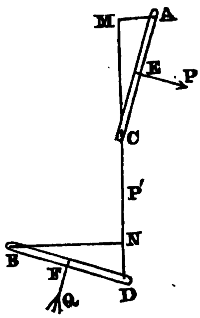
$$\frac{2^nP - P}{2 - 1}, \text{ or } (2^n - 1)P.$$

29. Prop. *Two levers moving in the same plane about their two fulcrums are connected by a link; to compare the moments which balance upon them.*

A link is a rod fastened at its extremity to another piece by a pin, so that it can turn through any angle.

Let AC, BD be the two levers, A, B their fulcrums, CD the link; P acting at an arm AE, and Q acting at an arm BF, the forces which balance: AM, BN perpendicular on CD.

Let P' be the force exerted by the link; this force balances each lever.



Hence, (Mech. 8.)

$$P.AE = P'.AM; \quad P'.BN = Q.BF.$$

Hence, $P.AE : Q.BF :: AM : BN$.

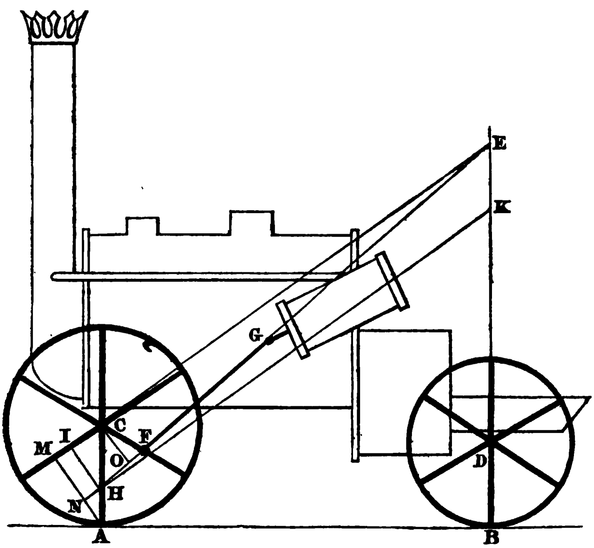
Cor. The moment of Q may be made very great by making AM very small; that is, by arranging the machine so that at the moment when the force is exerted, ACD is nearly a straight line.

30. A LOCOMOTIVE ENGINE is a machine which has a tendency to change its place by the effect of some action among its parts.

But in all such cases, some action upon external bodies is necessary, in order that the engine may travel; thus a row-boat advances by the action of the rowers upon the oars; but this effect requires the re-action of the water. The same is the case with a steam-boat. In like manner a carriage on a road, moved by steam, or any other power acting within it, requires the re-action of the road in order to advance.

31. PROP. *In a locomotive engine which is made to travel by turning a wheel on which it rests; the force required is the same as if the center of the wheel were fixed, and the resistance of the motion acted at the circumference of the wheel.*

The figure represents the original "Rocket" engine of Mr. Stephenson.



Let C, D be the axles of the fore and hind wheels of a locomotive engine, which rests on a road at the points A, B. And let a force act from some point of the carriage, by the rod GF, upon the crank CF, which turns with the wheel CA round the axle C. The rod exerts a certain action upon the crank at F, and an equal re-action upon the carriage at G: if FG meet CA and DB in H and E, the action on the wheel CA may be supposed to take place at E. Join CE, and draw HK parallel to CE, and let the equal action and re-action in the line GF or EH be represented by EH and HE.

The force EH is equivalent to KH, CH, of which the latter is supported and destroyed at A; the former is effective on the wheel CA.

The force HE is equivalent to KE, CE, of which the former diminishes the pressure at B, the latter draws C towards E.

Therefore the wheel CA is acted upon by two forces KH, CE, which tend to turn it about A. Also the resistance which the carriage experiences to its motion in the direction CD may be considered as a force acting at C perpendicular to CA. Let R be this force, and let ANM be perpendicular to CE, HK. Then, the forces and the resistance must balance each other; and by the property of the lever, if S be the force KH or CE,

$$R \times CA + S \times AN = S \times AM; \text{ or } R \times CA = S \times MN.$$

Let CO be drawn perpendicular from C upon EH, and HI upon CE; and let P be the pressure exerted by the rod GF, we then have

$$P : S :: EH : EC :: HI : CO :: MN : CO;$$

$$\text{therefore } P \times CO = S \times MN;$$

$$\text{and } R \times CA = P \times CO.$$

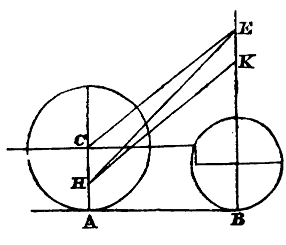
Hence the effect is the same as if C were fixt, and the force P acted by means of the rod GF and crank CF to produce motion in the wheel, in opposition to a force R acting at the circumference.

In order that the engine may be urged in the direction CD, the force which acts by means of the rod GF must be a pushing force while the line GF falls below C, and a pulling force when this line falls above C. If G be connected with the piston of a steam-engine, these two parts of the motion must correspond to the up and down stroke, or the backwards and forwards stroke.

The same reasoning which is here applied to locomotive steam-engines is applicable to all locomotive engines; for instance a row-boat. In the case of such a boat, the oar may be considered as a radius of a wheel, turning about the row-lock; the whole resistance of the water to the boat must then be considered as transferred to the shank of the oar, and balanced by the whole pull of the rower.

32. Prop. *In a locomotive engine which travels on fore and hind wheels, to find the vertical and horizontal pressure at the points of support.*

If the engine were not acted upon by the force which turns the wheel, it would rest at the points of support A, B; its whole weight being supported by these points, and this weight being divided between A and B, according to the position of the centre of gravity of the whole mass.



But the force P which acts in EH produces a pressure downwards at H and a pressure upwards at E. And it has been shewn in the last proposition, that if EH represent the

force P, CH and KE will represent the forces downwards at A, and upwards at B. Also, the forces CE and KH, along with R acting at C perpendicular to CA, balance each other upon the line CA, of which the fulcrum is A; and the pressure at A will be found by transferring all the forces to A. (Mech. 24.) But in this case, the forces CE and KH destroy each other. Therefore, the pressure at A, arising from the forces which turn the wheel, is R in the direction BA.

Cor. 1. In order that the engine may travel, there must be a re-action equal to R, acting at A in the direction AB. This may be produced by friction, or any other impediment to backward motion.

Cor. 2. The forces thus found are those requisite to *put* the machine in motion; or rather, to put it in the state bordering on motion. The forces requisite to *keep* it in motion, or to accelerate it, are of the same kind; but their magnitude is determined by the resistances which exist in the state of motion, not by statical considerations alone.

Cor. 3. We have $\sin ECH : \sin CEH :: EH : CH$,

$$CH = EH \frac{\sin CEH}{\sin ECH};$$

$$\text{or vertical pressure in } CA = P.\frac{\sin CEH}{\sin ECH};$$

this is the vertical pressure arising from the action of the working rod GF; and is to be added to that arising from the weight.

CHAPTER II.

THE PRINCIPLE OF VIRTUAL VELOCITIES.

33. By means of the Propositions contained in the preceding Chapter, we may determine the conditions of equilibrium of any machine whatever. But in many cases, the conditions of equilibrium of machines may be more conveniently determined by means of the principle of virtual velocities; which principle we shall therefore proceed to establish.

In a machine, a power or force acting on one part produces motion in another part, so as to do some proposed work; for instance, to raise a weight, or to bring two bodies in contact. The velocity of the part to which the power is applied, and of the part where the work is done, bear a certain relation to each other; and although in statics we consider the state of equilibrium, not the state of motion, the relation of the possible velocities of the parts of the machine, does, in reality, determine the conditions of equilibrium; and may often be the most convenient mode of deducing these conditions.

34. Prop. *The ratio of the virtual velocities of different points in a machine is the limit of the ratio of small spaces described by those parts.*

The virtual velocities are the velocities, or rather the *velocity ratios* (the ratios of the velocities), belonging to the different parts of a machine, in virtue of their connection with each other. If the machine were such that the spaces described by two points in the machine in the course of its motion bore a constant ratio to each other, this ratio would

also be the ratio of the velocities of the two points. But this ratio of the spaces described will in general be variable, and hence will not rightly express the velocity ratio. For if two points E, F, in a machine move into positions e, f, and again, into position e', f', the ratios Ee, Ff and Ee', Ff' being different; we cannot assume one of those ratios, rather than the other, to be the velocity ratio, and therefore neither of them can universally be so.

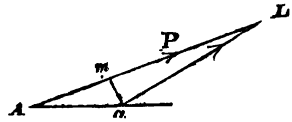
But the velocity ratio which belongs to the position in which the points are at E, F, must depend upon the position, and not upon any other. Hence it must be the *limit* of the ratio, Ee, Ff; namely the ratio to which this ratio tends indefinitely, as we take the points e, f, nearer and nearer to the points E, F.

Hence the velocity ratio, or the ratio of the virtual velocities, is the limit of the ratio of the spaces described.

COR. We are thus compelled in general to introduce the idea of a *limit* in order to reason concerning virtual velocities: we must also apply this idea in reasoning concerning other properties of our figures, as the properties of curve lines.

35. PROP. *If a point in a machine acted upon by a force in any direction, be moveable only in a direction making any angle with the force; the virtual velocity of the point estimated in the direction of the force is the virtual velocity in the direction of the motion multiplied into the cosine of the angle.*

Let Aa be a small space in the direction of the actual motion, AL the direction of the force, aL the direction of the force when the point is at a.



Take $Lm = La$, then the point A has moved nearer to L by the space Am. Hence the point has yielded to the force P through the space Am, supposing the force to act all the while in a direction passing through L. But the space through which the body has yielded to the force is the space which the body has described, estimated in the direction of the force. Hence Am is the space described estimated in the direction of the force P, supposing the force to act in the same direction all the while. But these suppositions,—that the force acts all the while through L, and that it acts all the while in the same direction,—are both true, if we take the limit, and suppose Aa indefinitely small. Hence in that case, Aa being the virtual velocity of A in the direction Aa, Am is the virtual velocity of A estimated in the direction of the force P.

But since $Lm = La$, angle $Lma = Lam$,
and $2Lma =$ supplement of aLm; that is, at the limit, $2Lma = 2$ right angles.

Hence at the limit, Ama is a right angle, and

$$Am = Aa \cos LAa.$$

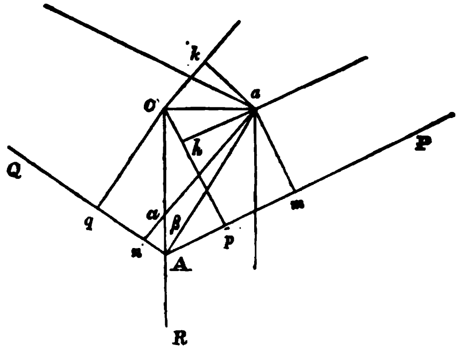
Cor. If a point A move through a *small* space Aa, and if from a a perpendicular am be drawn upon any line AP, Am is the virtual velocity in the direction AP.

By speaking of *small* spaces, we imply that the *limiting ratios* are to be taken.

36. Prop. *If three forces* P, Q, R *keep a point* A *in equilibrium; and, the point being moved through any small space, if* p, q, r *be the virtual velocities estimated in the direction of the forces* P, Q, R *respectively (those being reckoned negative in which the point moves opposite to the force)*;

$$\mathrm{Pp} + \mathrm{Qq} + \mathrm{Rr} = 0.$$

Let the point A move through a small space Aa, and let am, an, ao be drawn perpendicular upon the directions



of the forces, AP, AQ, AR. Then Am, An, Ao will be the virtual velocities in the directions of the forces P, Q, R, and are therefore p, q, r respectively.

Let op, oq, be perpendicular upon AP, AQ; and ah, ak perpendicular upon op, oq.

Let the angles QAo, PAo be respectively α, β; then since the triangles Aop, Aoq are right-angled, $Ap = r\cos\beta$, $Aq = r\cos\alpha$. Hence

$$mp = p - r\cos\beta,\ nq = r\cos\alpha - q.$$

Now the triangles aoh, aok are right-angled; and angle aoh = complement of angle Aop (for Aoa is a right angle) $=\beta$. Similarly angle aok = complement of $Aoq = \alpha$.

$$\text{Hence } ao\sin\beta = ah = mp = p - r\cos\beta,$$

$$ao\sin\alpha = ak = nq = r\cos\alpha - q.$$

$$\text{Whence } \frac{p - r\cos\beta}{r\cos\alpha - q} = \frac{\sin\beta}{\sin\alpha};$$

$$\text{or, } p\sin\alpha - r\sin\alpha\cos\beta = r\cos\alpha\sin\beta - q\sin\beta;$$

$$\text{or, } p\sin\alpha + q\sin\beta = r\sin\alpha\cos\beta + r\cos\alpha\sin\beta;$$

$$\text{or, } p\sin\alpha + q\sin\beta = r\sin(\alpha+\beta).$$

Hence multiplying by R,

$$Rp\sin\alpha + Rq\sin\beta = Rr\sin(\alpha+\beta).$$

But (Mech. 17.)

$P : R :: \sin QAR : \sin PAQ :: \sin QAo ; \sin PAQ :: \sin \alpha : \sin (\alpha + \beta)$; hence $R \sin \alpha = P \sin (\alpha + \beta)$. Similarly, $R \sin \beta = Q \sin (\alpha + \beta)$. Therefore substituting, and omitting $\sin (\alpha + \beta)$, which occurs in every term,

$$Pp + Qq = Rr, \text{ and } Pp + Qq - Rr = 0.$$

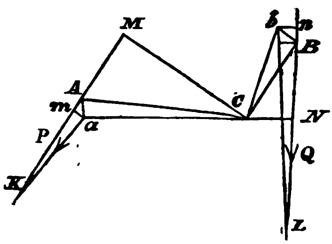
But we must put $-r$ instead of r, because the motion of the point A is opposite to the force R. Hence

$$Pp + Qq + Rr = 0.$$

This is the *Principle of virtual velocities.* It is here proved for three forces acting upon a point; but the same proposition is true of any number of forces acting upon a machine, as we shall see.

37. PROP. *The Principle of Virtual Velocities is true in any Lever.*

Let ACB be any lever kept in equilibrium by the forces P, Q; and let the lever be moved through a small angle into the position aCb. Let am, bn, be drawn perpendicular upon AP, BQ: then Am, Bn are the virtual velocities, p, q, corresponding to the forces P, Q. Let also CM, CN be perpendicular upon AP, BQ.



The motion being very small, CAa is very nearly a right angle, and will be a right angle at the limit. Hence the angle aAm is the complement of CAM, and is therefore equal to ACM. Hence the right-angled triangles aAm and ACM are similar; and $Am : CM :: Aa : AC$. In like manner the triangles bBn and BCN are similar,

$$\text{and } Bn : CN :: Bb : BC.$$

But since the angles ACa, BCb are equal,

$$Aa : AC :: Bb : BC.$$

Hence we have

$$Am : CM :: Bn : CN,$$
$$\text{or } p : q :: CM : CN.$$

But by the property of the lever (Mech. 7.)

$$CM : CN :: Q : P.$$

Hence $p : q :: Q : P$;

$$\text{and } Pp = Qq, \text{ or } Pp - Qq = 0;$$

the opposite motion *not* being denoted by a negative quantity.

38 PROP. *The Principle of Virtual Velocities is true of two forces* P, Q *acting upon a piece constrained by sliding.*

As we have seen in Art. 8, the two forces P, Q which balance on such a piece as is mentioned in the enunciation, meet in a point; and the remaining force which is requisite for the equilibrium of the piece is supplied by the constraining force, and is perpendicular to the direction of sliding. But in that direction no motion is possible. Hence, $r = 0$, and the equation of Art. 36 becomes

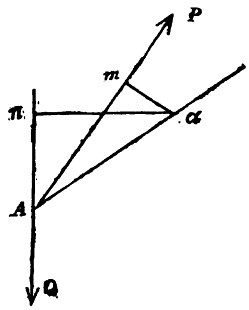
$$Pp + Qq = 0;$$

which is the principle of virtual velocities for two forces.

If p be positive, q will be negative; that is, if the body in moving yields to one force, it moves opposite to the other.

On the inclined plane this proposition may be proved by an independent method.

39. PROP. *On the Inclined Plane the principle of virtual velocities is true.*



Let Aa be a small space described by any point of a heavy body moving upon an inclined plane, its weight Q being balanced by a force P. Let am be perpendicular on AP, an on QA. Hence $Am = p$, $An = q$.

Also $PAa = \alpha$, $aAn = \beta$, as in (Art. 11.)

And hence $Aa \,.\cos\alpha = Am = p$, $Aa\,.\cos\beta = An = q$,

$$\frac{\cos\alpha}{\cos\beta} = \frac{p}{q};\ q\cos\alpha = p\cos\beta,\ p\cos\beta - q\cos\alpha = 0;$$

$$\text{hence } Qp\cos\beta - Qq\cos\alpha = 0.$$

But (Art. 10) in this case, $Q\cos\beta = P\cos\alpha$.

Hence substituting, $Pp\cos\alpha - Qq\cos\alpha = 0$;

$$\text{or } Pp - Qq = 0;$$

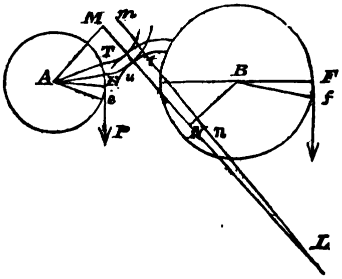
which is the principle of virtual velocities in this case, the opposite motion *not* being denoted by a negative quantity.

We shall now prove the Principle for a machine composed of two pieces acting on each other by contact.

40. PROP. *The principle of virtual velocities is true of two forces* P, Q, *which balance each other by means of two wheels acting on each other by contact.*

That is, if two forces P, Q, balance each other, by acting upon two toothed wheels, respectively, acting perpendicularly at radii AE, BF, and if a small motion be given to the machine, so that the points E, F describe small spaces Ee, Ff, we shall have $P\,.\,Ee = Q\,.\,Ff$.

Let T be the point of contact of the wheels, and let



MTN be perpendicular to the surface of contact, and therefore the line of action. By the small motion, let the point

of contact become t, and the line of action mtn; and let MN, mn, produced meet in L.

The action of each wheel upon the other may be considered as equal and opposite pressures acting in the line MTN at T. Let R represent the pressure which the wheel B thus exerts upon the wheel A. Then the force R may be supposed to act at L, in the direction LT. And while the point of contact moves from T to t, the force R may still be supposed to act at L, and, at the limit, to be constant in its direction; and every point of the line TL will move in the direction of its length through a space $LT - Lt$, that is, at the limit, Tu; tu being perpendicular to LT. Hence by Art. 37, $P . Ee = R . Tu$. In like manner, since R is also the force which the wheel A exerts upon the wheel B, and since R may be supposed to act at L in the direction TL, we have $Q . Ff = R . Tu$. Wherefore $P . Ee = Q . Ff$.

Cor. 1. If p, q be the virtual velocities of the points E, F, $Pp = Qq$, or $Pp - Qq = 0$.

Cor. 2. If MN, mn be rigorously parallel, the proposition is still true; for this is the case to which we approach as a limit, when the point L is supposed to be at an infinite distance.

41. Prop. *If two forces* P, Q *balance each other by acting upon a cam and a sliding piece respectively, the force* P *acting at a radius* AE; *and if a small motion be given to the machine, so that the point* E *describes a small space* Ee, *and every point in the cam a small space* Ff; *we shall have* P.Ee = Q.Ff.

This machine is identical with a pair of toothed wheels in which the radius of the second wheel B is infinite. For in this case, the motion of any point F of the wheel B

would be in a constant direction, being always perpendicular to the infinite radius.

Hence by the last Article

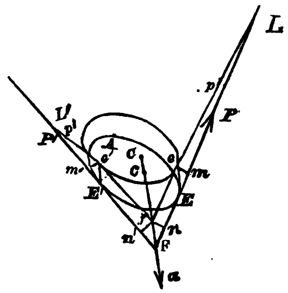
$$P.Ee = Q.Ff.$$

Cor. In the same manner if two forces P, Q balance each other by acting upon two sliding pieces, and if a small motion be given to the machine, so that the two pieces describe small spaces Ee, Ff respectively, we shall have

$$P.Ee = Q.Ff.$$

42. Prop. *The principle of virtual velocities is true of two forces which balance by means of a band wrapping round a piece of any form.*

Let A be the piece; $PEE'P'$, the wrapping band; P, the force which pulls the band; Q, the force which acts on the piece; F, the point in which PE and $P'E'$ meet.



Let C be a point in A; and let the piece A be moved parallel to itself through a small space Cc in opposition to the force Q, and let the wrapping band assume the position $pee'p'$. Then at the limit we may suppose ep, $e'p'$ to be parallel to EP, $E'P'$, and the points of contact E, E' will be moved through spaces Ee, $E'e'$, equal and parallel to Cc. Also the point F will be moved through a space Ff equal to Cc, and in the line parallel to Ee, $E'e'$.

Draw em, fn perpendicular upon FP; $e'm'$, $f'n'$ perpendicular upon FP'. If L be a fixed point in EP, by the small motion of the piece A the point of contact E is brought nearer to L by a space Em. In like manner the point of contact E' is brought nearer to a fixt point L', in the line $E'P'$, by a space $E'm'$. Hence by the small motion of the

piece A, a portion of the band is liberated equal to $Em + E'm'$, and supposing L' to remain fixt, a point in the band EP will move through this space in the direction of the force P, which pulls it. But $Em = Fn$, and $E'm' = Fn'$. And by Art. 24, the angles CFn, CFn' are equal, whence $Fn = Fn'$; hence p, q, the virtual velocities in the directions of the forces P, Q, are as $2Fn : Ff$. But by Art. 24, $Q = 2P \cos CFE$; whence $2P . Ff \cos CFE - Q . Ff = 0$; that is, $P . 2Fn - Q . Ff = 0$, or $Pp - Qq = 0$.

43. Prop. *If in any machine two pieces in two simple machines be connected by a link, the principle of virtual velocities is true.*

Let a simple machine in which the forces are P, Q, be connected by a link with another simple machine in which the forces are P', Q'; and let p, q, p', q' be the virtual velocities of the points; then we have

$$Pp + Qq = 0, \quad P'p' + Q'q' = 0,$$

the motions opposite the forces being reckoned negative. But the forces Q, P' are the opposite forces exerted by the link in the direction of its length, on the two simple machines, and are therefore equal; and the virtual velocities q, p' are the velocities of the link in the direction of its length, and are in the same direction. Hence if one of these virtual velocities be in the direction of its force, the other will be opposite to its force. Therefore

$$P' = Q, \text{ and } p' = -q.$$

Hence $Pp + Qq = 0$, $-Qq + Q'q' = 0$; and adding

$$Pp + Q'q' = 0.$$

Cor. The same is true if two pieces be connected by a cord, except that the cord can act by pulling only, not by pushing.

44. Prop. *If two forces act upon any piece in a machine, the principle of virtual velocities is still true.*

Let two forces, R, S, act upon any piece, instead of a single force P, to balance Q; and let r, s, q be the virtual velocities. The two forces R, S may be conceived to meet in a point, and Q to be their resultant. Hence R, S, $-Q$, are three forces which are in equilibrium at a point, and by Art. 36,

$$Rr + Ss - Qq = 0.$$

But, by the preceding propositions,

$$Pp + Qq = 0.$$

Hence, adding, $Pp + Rr + Ss = 0$.

Cor. If any number of forces act upon a piece in a simple machine, the principle of virtual velocities is still true.

Let three forces, R, S, T keep a piece in equilibrium instead of a single force Q; let R, S be equivalent to a single force U; and let r, s, t, q, u, be the virtual velocities.

Since U is equivalent to R, S, $Rr + Ss - Uu = 0$.

Since Q is equivalent to T, U, $Tt + Uu - Qq = 0$;

and, by the simple machine, $Pp + Qq = 0$.

Hence, adding, $Pp + Rr + Ss + Tt = 0$.

And the same kind of proof may be extended to any number of forces.

45. Prop. *If any forces* P, Q, R &c. *act on points in any machine, simple or compound, and keep each other in equilibrium, and if* p, q, r, &c., *be the virtual velocities of the points at which they act, we shall have,*

$$Pp + Qq + Rr + \&c. = 0.$$

Any machine whatever is composed of pieces which transmit the action of forces, in some of the ways mentioned

in Articles 37 to 42, each of these pieces being acted upon by one or more forces. If some of the intermediate pieces act upon each other by contact or by links, the forces thus exerted multiplied by their virtual velocities for the two connected pieces, and added together, will vanish, the positive and the negative forces destroying each other. Let P act on a piece A, which acts on a piece B: and let B be acted on by a force Q, and connected with a piece C, on which it exerts a force Q'; let C be acted on by a force R, and connected with two pieces D, E, on which it exerts forces R', R''; and let D, E be kept in equilibrium by forces S, T. Also let p, q, q', r, r', r'', s, t, represent the corresponding virtual velocities. Then, by Articles 37 to 42,

$$\text{by the action of } A \text{ on } B, \quad Pp + Qq + Q'q' = 0:$$

$$\text{by the action of } B \text{ on } C, \quad -Q'q' + Rr + R'r' + R''r'' = 0:$$

$$\text{by the action of } C \text{ on } D, \quad -R'r' + Ss = 0:$$

$$\text{by the action of } C \text{ on } E, \quad -R''r'' + Tt = 0.$$

Hence, adding, $Pp + Qq + Rr + Ss + Tt = 0$.

46. Prop. *The Principle of virtual velocities is true in the case of Locomotive Engines.*

Such engines are included in the preceding demonstration: but the Proposition may be proved independently by reference to Article 31.

In the figure there given, $R \times CA = P \times CO$, where P is the pressure exerted by the rod GH, and R the resistance to motion which is supposed to be just balanced. If the rod GF move through any small space p in the direction of its length, C being supposed to be fixt, and the point A through a space r, the case is that of a lever with a fixt center; wherefore $Pp = Rr$.

Now if instead of C being fixt, the wheel roll at A, C will move forwards by a space r, and the rod GF will move through the space p in the direction of its length, as before. Therefore we have still, $Pp = Rr$.

If the locomotive engine be driven by a *piston* working backwards and forwards, and acting on the wheel by means of the connecting rod GF, the piston moves in the direction of its length, but the connecting rod GF does not move in the direction of its length. Let, as before, P be the pressure of the connecting rod in its own direction, p its virtual velocity in the direction of its length. Q the pressure on the piston, q its virtual velocity: therefore by Art. 38, $Pp = Qq$. Also as before, $Pp = Rr$: therefore $Qq = Rr$.

Cor. Let Q be the *mean* pressure on the piston, l the length of the stroke, D the diameter of the driving wheel. Then, while the piston moves up and down through $2l$, the wheel makes one revolution, and C travels through πD. And the sum of all the elements Qq in this time is $Q \times 2l$, and the sum of all the elements Rr is $R \times \pi D$. Therefore $\pi DR = 2Ql$.

47. Prop. *If any body or system of points acted upon by gravity only be in equilibrium, and if a small motion be given to the system: the center of gravity will move horizontally, or will rest.*

When a connected system of heavy particles are moved into various positions, in general their center of gravity describes a curve. And it is to be proved that the position of equilibrium will be that in which the center of gravity is at the highest or lowest point of this curve.

Let P, Q, R be the weights of any heavy connected particles; and let p, q, r be small vertical spaces described

by the points P, Q, R, when a small motion is given to the system: those which take place upwards being negative. Therefore by Art. 45, $Pp + Qq + Rr = 0$.

Let x, y, z be the vertical depths of P, Q, R below a horizontal line, the position of equilibrium: and $x + p$, $y + q$, $z + r$, their depths below the same horizontal line when the small motion has taken place in the system: h and h' the depths of the center of gravity below the same horizontal line in the two cases.

$$\text{Then } h = \frac{Px + Qy + Rz}{P + Q + R}$$

$$h' = \frac{Px + Pp + Qy + Qq + Rz + Rr}{P + Q + R} = h;$$

because $Pp + Qq + Rr = 0$.

Hence in the two positions the center of gravity is at the same depth below the horizontal line; and therefore by the small motion, the center of gravity moves horizontally.

Cor. Hence in general, in the position of equilibrium, the center of gravity is at the highest or lowest point of the curve which it can describe.

CHAPTER III.

STATICAL COUPLES.

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48. Def. *If two forces, equal and opposite, but not in the same straight line, act upon a rigid body, they are termed a* STATICAL COUPLE.

Such a couple of forces possesses very curious properties. They cannot be balanced by any single force, nor by any number of forces which are reducible to a single force: but they will be balanced by any one of an infinite number of other couples. A couple may be transferred to any distance, in its own plane or in any parallel plane, and turned through any angle, without altering its effect. A couple tends to produce rotation, but the axis about which this rotation should take place is perfectly undetermined, and the tendency may be counteracted by a tendency to an opposite rotation about an axis at any distance from the other.

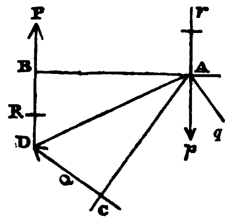
We shall proceed to prove some of these properties, and to apply them to certain mechanical problems.

Def. *The straight line perpendicular to the directions of the two forces, and terminated by them, is called the* ARM *of the couple.*

Def. *The product of one of the forces into the arm, is called the* MOMENT *of the couple.*

49. Prop. *A couple may be turned about either of the extremities of the arm through any angle without altering its effect.*

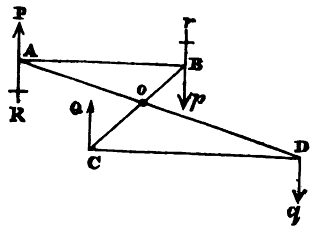
Let P, p be a couple with the arm AB; and let the arm assume the position AC, and the forces be now denoted by Q, q; the effect is not altered.



Let R, r be forces equal and directly opposite respectively to P, p; therefore R, r balance P, p. But if R, Q, meet in D, the forces R, r, Q, q balance: for R, Q (being equal) are equivalent to a force in AD, which evidently bisects the angle BDC (AB, AC being equal); and r, q (being equal) are equivalent to a force in DA, which produced, bisects the angle rAq for the same reason. And the forces in AD and DA are equal, because R, Q, and r, q are equal. Hence the forces R, r, Q, q balance; that is, R, r balance Q, q; but R, r balance P, p: therefore Q, q produce the same effect as P, p.

50. PROP. *A couple may be transferred, parallel to itself, to any distance in its own plane, without altering its effect.*

Let P, p be a couple with the arm AB; and let it be transferred to Q, q with the equal and parallel arm CD; the effect is not altered.



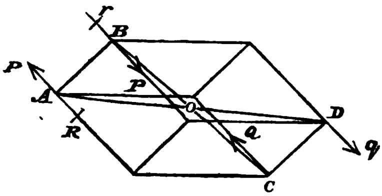
Let R, r be forces equal and directly opposite respectively to P, p; therefore R, r balance P, p.

But the four, R, r, Q, q, balance: for join AD, BC; these bisect each other in O, as is easily shewn. The two, R, q which are equal, produce at O a pressure parallel to them and equal to their sum; and the two, Q, r which are equal, produce at O a pressure in the opposite direction equal to their sum; and these sums are equal, because

Q, R and q, r are equal. Therefore the four, R, r, Q, q, balance: that is, R, r balance Q, q. But R, r balance P, p; therefore Q, q produces the same effect as P, p.

51. PROP. *A couple may be transferred to any plane parallel to its own plane without altering its effect.*

Let P, p be a couple with the arm AB; and let it be



transferred to Q, q with the equal arm CD; the effect is not altered.

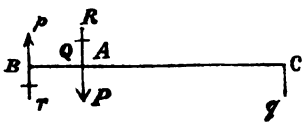
Let R, r be forces equal and directly opposite to P, p; therefore R, r balance P, p. But the four, R, r, Q, q, balance: for join AD, BC; these bisect each other in O, as is easily shewn. The two R, q, which are equal, produce at O a pressure parallel to them and equal to their sum; and the two Q, r, which are equal, produce at Q a pressure in the opposite direction equal to their sum; and these sums are equal, because Q, R and q, r are equal. Therefore the four, R, r, Q, q, balance; that is, R, r balance Q, q. But R, r balance P, p: hence Q, q produce the same effect as P, p.

COR. Hence a couple may be transferred to any part of any plane parallel to its own plane, and turned through any angle, without altering its effect.

52. PROP. *Two couples in the same, or parallel planes, which have equal moments, will have the same effect.*

If necessary, let one couple be transferred into the same plane with the other (Art. 51), transferred in that

plane till the extremities of their arms coincide (Art. 50), and turned about that extremity till the arms are in the same straight line (Art. 49). Let P, p with the arm AB, and Q, q with the arm AC, be these couples: and let the moments be equal, that is, $p \times AB = q \times AC$. Let R, r be forces equal and directly opposite to P, p; therefore R, r balance P, p. But the four, R, r, Q, q, balance; for since $r = p$, $r \times AB = q \times AC$: therefore by the property of the lever, q and r balance on A, and produce there a pressure $q + r$; and this pressure is balanced by the forces Q, R, which act at A in the opposite direction. Therefore the four, R, r, Q, q, balance; that is, R, r balance Q, q. But R, r balance P, p; therefore Q, q produce the same effect as P, p.



DEF. *The* AXIS *of a statical couple is a line perpendicular to the plane in which are the two forces, and the arm at which they act.*

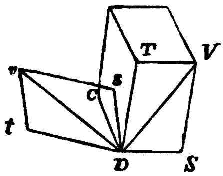
It appears by the last Article, that so long as the axis continues parallel to itself, and the moment is the same, the effect of the couple is the same.

We may *represent* a couple in direction and magnitude by a line drawn in the direction of its axis and proportional to the moment of the couple.

53. PROP. *If two adjacent sides of a parallelogram represent in direction and magnitude two couples which act upon the same rigid body, the resulting effect will be a single couple represented in direction and magnitude by the diagonal of the parallelogram.*

Let DS and DT represent in direction and magnitude two couples which act upon a rigid body. Each of them may be supposed to act at any arm, provided the moment

continue the same; put DC perpendicular to the plane SDT, the arm for both the one couple and the other.



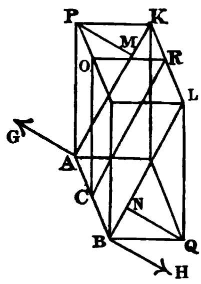
In the plane SDT, let Ds be perpendicular and equal to DS, Dt perpendicular and equal to DT. Then the couple of which DS is the axis, is in the plane CDs, and the couple of which DT is the axis, is in the plane CDt: and these couples may consist respectively of two forces, acting in Ds, and at C, equal and opposite to Ds, and two forces acting in Dt, and at C, equal and opposite to Dt: the moments of these pairs of forces on the arm CD being as Ds, Dt. Therefore the forces themselves may be represented by Ds, Dt, and by equal and opposite lines at C. Complete the parallelogram $Dsvt$; and the forces are equivalent to a force Dv, and to an equal and opposite force at C; that is, to a couple of which the moment is as Dv, and the plane is cDv. But if we complete the parallelogram $DSVT$, it will be in all respects equal to $Dsvt$. Hence angle $VDv = SDs$, a right angle. Also CD is perpendicular to the plane SDT, therefore to DV; therefore VDc is a right angle. Therefore VD is perpendicular to the plane CDv, and therefore VD is the axis of the resulting couple. And the force is as DV with the arm CD, that is, it is represented by DV.

Cor. Hence all the consequences of the composition of forces are true of the composition of couples, represented as in this Article.

54. Prop. *Any forces acting upon a rigid body are equivalent to a single force, and to a couple, of which the axis is in the direction of the single force.*

It may be shewn (Mech. 38), that any forces acting upon a rigid body are equivalent to two forces; one, a

force P, acting in a certain direction AP (x); and the other, a force Q, acting in the direction BQ, in a plane, (xy), perpendicular to this direction. Let AP, BQ represent the forces; and let AB be perpendicular to BQ. Divide AB in C, so that



$$AC : BC :: Q^2 : P^2;$$

draw CO parallel and equal to AP; and OR, parallel and equal to BQ; and join CR. The forces which are equivalent to P and Q are equivalent to a single force acting in the line CR, and to a couple acting in a plane perpendicular to this line.

Complete the parallelopiped, of which the edges are PA, AB, BQ; and COR will be a section of this parallelopiped, made by a plane parallel to the ends in which AP or BQ are. Also let AK, BL be the diagonals of the ends of the parallelopiped; and let lines PM, QN be perpendicular upon AK, BL.

The force AP is equivalent to the forces AM, MP, both acting at A; and the force BQ is equivalent to the forces BN, NQ, acting at B. Hence the forces P, Q are equivalent to forces AM, BN acting at A, B; and also to forces MP, NQ acting at A, B.

Now AP, PK represent P, Q respectively.

$$\text{Hence } AK \text{ is } \sqrt{P^2 + Q^2}.$$

$$\text{Also } AK : AP :: AP : AM, \text{ or } \sqrt{P^2 + Q^2} : P :: P : AM.$$

$$\text{And } AK : KP :: AP : PM, \text{ or } \sqrt{P^2 + Q^2} : Q :: P : MP.$$

$$\text{Hence force } AM = \frac{P^2}{\sqrt{P^2 + Q^2}}; \text{ and force } MP = \frac{PQ}{\sqrt{P^2 + Q^2}}.$$

In like manner, because BQ, QL represent Q, P, we have force $BN = \frac{Q^2}{\sqrt{P^2+Q^2}}$; force $NQ = \frac{PQ}{\sqrt{P^2+Q^2}}$.

The parallel forces AM, BN, are as $P^2 : Q^2$; that is, by construction, as $BC : AC$; hence they are equivalent to a single force, acting at C, parallel to these forces, and equal to their sum: that is, to a force $= \frac{P^2+Q^2}{\sqrt{P^2+Q^2}} = \sqrt{P^2+Q^2}$.

The forces MP, NQ, or AG, BH, are equal, each being

$$= \frac{PQ}{\sqrt{P^2+Q^2}};$$

and they act at right angles to AB, in the same plane perpendicular to AK, BL, or CR, in opposite directions at its two extremities: that is, they are a couple of which the axis is any line parallel to CR.

Cor. 1. The single resulting force, CR, is the same in magnitude and direction as the resultant of all the forces transferred to a single point, retaining their parallelism.

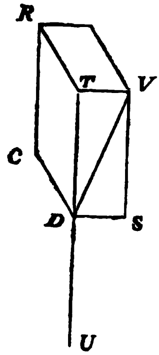
Cor. 2. If $AB = a$, the moment of the couple of which the axis is CR, is

$$= \frac{aPQ}{\sqrt{P^2+Q^2}}.$$

55. Prop. *When any forces act upon a rigid body, to reduce them to a single force, acting through a given point, and to a couple.*

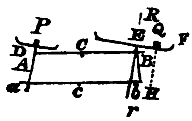
By last Article, let the forces be reduced to a single force R, acting in the direction CR, and to a couple of which the axis is CR, and the moment aS. Let D be the given point; let the couple act in the plane passing through D; and let the factor S be taken such that $CD = a$.

At D let two equal and opposite forces, T, U, each equal to R, act, in the direction DT parallel to CR, and DU the opposite direction. These two forces balance each other, and therefore do not alter the resultants of the former forces. But these forces being introduced, the body is acted upon by a single force T; a couple R, U acting at an arm CD; and the couple aS: that is, a single force R, acting in the line DT; a couple aR of which the axis is DS, perpendicular to the plane RCD; and a couple aS, of which the axis is DT. Take DT, DS in the proportion of R, S; and complete the rectangular parallelogram $DSVT$. The two couples aR, aS, of which the moments are represented in magnitude by DT, DS, are together equivalent to a couple represented in magnitude by DV, and of which the axis is DV. But $DV = \sqrt{R^2 + S^2}$; hence the couples are equivalent to a couple of which the moment is $a\sqrt{R^2 + S^2}$, and the axis DV. Therefore all the forces are equivalent to a single force R, acting in DT, and a couple of which the moment is $a\sqrt{R^2 + S^2}$, and the axis DV.



COR. 1. Hence we may reduce all the forces which act upon a rigid body to a single force, acting at the center of gravity, and a couple of given moment and axis: namely, by taking the center of gravity for the given point D.

56. The following Examples may serve to shew the application of these principles.



Let AB, ab be two perfectly equal bars, turning at their middle points about pivots C, c, which are in the same vertical line; and connected by means of pivots A, a, B, b, with two other equal bars Aa, Bb. Then Aa, Bb will always be vertical,

as is easily shewn. Let horizontal scale-pans be fixed to the vertical rods at D and E, and we have a balance which is in common use.

If instead of scale-pans, we have horizontal rods fixed to any part of the vertical rods Aa, Bb, and weights, P, Q, attached to these rods, we have a balance possessing paradoxical properties, and called, from the name of its inventor, *Roberval's Balance.*

The paradox consists in this, that equal weights, P, Q, will balance each other at unequal distances from the center of motion C.

The common balance above described exhibits the same paradox, for the object weighed, Q, may be placed in any part of the scale-pan EF, and its weight will still be equal to the weight P, in the other scale-pan. Therefore for the sake of distinction, we shall still call this Roberval's Balance: and we proceed to prove that it is a true balance.

57. Prop. *Equal weights, at unequal distances from the balance, are in equilibrium in Roberval's Balance.*

Let P, Q be equal weights in the balance, P acting directly on the vertical rod DA, and Q acting at any distance EQ, from the vertical rod EB. Suppose equilibrium to subsist.

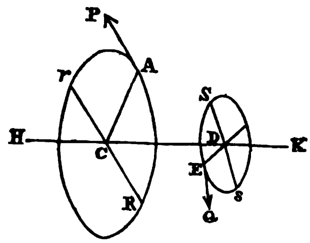
Introduce at E two equal and opposite forces R, r, acting vertically upwards and downwards, each equal to Q: the equilibrium will not be disturbed. But we have thus two equal forces P, r acting in the vertical lines DA, EB, and a pair Q, R with the arm EQ. The former two forces produce a vertical pressure $P + Q$ upon C, c, with no tendency of the bars to turn round the pivots. The couple QR acts upon the vertical rod EB; and without altering its effect, in the condition of equilibrium, it may have the extremity of its arm transferred to B (Art. 50), and may be turned round B, so that its arm coincides with BH, perpendicular to BC

(Art. 49); and may have the length of its arm changed to BH, provided the moment of the couple continues the same (Art. 52); that is, if for Q, R we substitute two forces S, T, each equal to $Q \times \frac{EQ}{BH}$. But when the couple is thus transferred and transformed, we have, as its result in the system, two equal forces S, T, acting respectively in the lines CB, bc. Now these forces will produce pressure on the pivots C, c, with no tendency of the bars to rotation. Hence when P, Q are equal, there is no tendency to rotation; that is, P and Q balance.

Cor. If the weight P, as well as the weight Q, were at any distance from the vertical rod DA, the equilibrium would still subsist.

58. Prop. *When any two forces balance each other on the wheel and axle, the pressures upon the points of support are the same as if the forces were transferred to the points where the planes perpendicular to the axis meet the axis.*

Let P, Q be two forces acting on the wheel and axle perpendicularly to the arms CA, DE; balancing about the



axis HK: the pressures at the points H, K, are the same as if the forces P, Q were transferred to the points C, D.

Introduce at the point C two forces R, r, equal and opposite, and each equal to P. Also introduce at the point D

two forces equal and opposite, and each equal to Q. We have then two couples, P, R, with the arm AC; Q, S, with the arm DC; and besides, two forces r, s, acting on the axis at C, D. But since P, Q balance on the wheel and axle, $P \times AC = Q \times DE$; therefore the moments of the two couples are equal, and they will produce equal effects; and since they tend to turn opposite ways, will balance each other. Therefore the only forces which produce pressure on the axis are the forces r, s; that is, the pressure upon the axis is the same as if the forces P, Q, keeping their parallelism, were transferred to the points of the axis opposite to them respectively.

59. In this example we may calculate the pressures at H and K as follows:

Let P make an angle α with a vertical line, and Q an angle β. Therefore P is equivalent to a vertical force $P \cos \alpha$, and a horizontal force $P \sin \alpha$; and in like manner Q is equivalent to a vertical force $Q \cos \beta$, and a horizontal force $Q \sin \beta$. Let H, K represent the forces at the points H, K, respectively, and let ϕ, ψ be the angles which H, K respectively make with the vertical lines. Therefore the vertical forces at H, K, are $H \cos \phi$, $K \cos \psi$, and the horizontal forces at H, K, are $H \sin \phi$, $K \sin \psi$. But all the vertical forces which act upon the axis HK must balance each other by the property of the lever, as must also the horizontal forces. Hence, (Mech. 22,) considering H as the fulcrum,

$$P \cos \alpha . HC + Q \cos \beta . HD = K \cos \psi . HK,$$

$$P \sin \alpha . HC + Q \sin \beta . HD = K \sin \psi . HK.$$

$$\text{Hence, } \tan \psi = \frac{P \sin \alpha . HC + Q . \sin \beta . HD}{P \cos \alpha . HC + Q . \cos \beta . HE}.$$

$$K^2 = \left(P\cos\alpha\,\frac{HC}{HK} + Q\cos\beta\,.\,\frac{HD}{HK}\right)^2$$

$$+ \left(P\sin\alpha\,\frac{HC}{HK} + Q\cos\beta\,.\,\frac{HD}{HK}\right)^2;$$

whence the magnitude and direction of K are determined; and the magnitude and direction of H may be determined in the same manner.

For example, let Q be vertical: P horizontal; HC, CD, DK all equal. Then

$$HD = 2HC, \quad HK = 3HC;$$

$$\tan\psi = \frac{P}{2Q}; \quad K = \frac{\sqrt{(P^2 + 4Q^2)}}{3},$$

$$\tan\phi = \frac{2P}{Q}; \quad H = \frac{\sqrt{(4P^2 + Q^2)}}{3}.$$

CHAPTER IV.

THE EQUILIBRIUM OF STRUCTURES.

SECT. I. THE EQUILIBRIUM OF FRAMES.

60. STRUCTURES are of various kinds, as *Frames*, which have their parts connected by pins or mortises; and *Arches*, in which the parts are connected only by contact.

The pressure exerted on the parts of structures by the forces which act, and resisted by their parts, is called *stress*.

In this chapter we consider the conditions of structures, and the stress exerted upon their parts, which may be deduced from the principles of Statics.

61. Structures are of various forms and uses, as roofs, floors, bridges; and of various materials, as wood, stone, iron.

Wooden and iron structures are composed principally of longitudinal pieces, as beams, posts, &c. which preserve the form of the structure by resisting extension, compression, and flexure. Some of these pieces have different appellations, according to their different offices.

62. A *tie* is a piece in a structure, fixt at its two extremities, and resisting extension.

A *strut* is a piece fixt at its extremities, and resisting compression.

A *brace* is a term which includes both a strut and a tie.

A frame connected by braces, so that it cannot change its form without rupture of the material, is said to be *braced*. Such a frame is also called a *truss*, and said to be *trussed*.

Braces may be conceived to be fixt at each extremity by one pin, in which case they have no force to resist

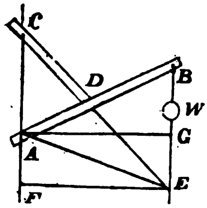
motion about that point; or they may be fixt by ***two pins***, so that turning is prevented. Or turning may be prevented by the end of one piece being tightly inserted into a hole made in the other piece: the end and the hole are called a ***tenon*** and a ***mortise***, and the piece is said to be ***mortised***.

Sometimes a strut is prevented from sliding along one of the pieces which it connects by a projection, which is called a ***shoulder***.

A frame which alters its position by changing the amount of its angles is said ***to rack***.

A roof is usually supported by ***roof-trusses*** placed at certain intervals. One of the most common cases is that in which each roof-truss consists of two ***rafters***, sloping from the roof-tree to the ***wall-plate***, and a horizontal ***tie-beam*** connecting the feet of the rafters. These rafters are sometimes called ***principals***.

63. ***When a loaded beam fixt at one end is supported by a tie, the points of support of the beam and the tie being in the same vertical line: to find the magnitude and direction of the stress at the points of support.***



Let AB be the loaded beam, held by a fixt pin at A, and supporting the load W at B; CD the tie held by a fixt pin at C, (CA being vertical) and fastened to the beam by a pin D.

Let CD produced meet in E, the vertical line through B; join AE.

The load acting in the vertical line BE is supported by the force at C, which acts in the line EC, and by a force at A, which must act in the line EA, because the three forces must meet in one point (Mech. 23). Also the weight of the load acts parallel to CA. Hence the three forces which keep the load in equilibrium, will be as the lines EC, CA;

AE; and therefore, CA representing the load W, EC and AE will represent the forces exerted *by* the points C and A, and CE and EA will represent the forces exerted *upon* the points C and A.

Hence the pressure upon C is $W.\dfrac{CE}{CA}$, in the direction CE; and the pressure exerted upon A is $W.\dfrac{EA}{CA}$, in the direction EA.

COR. 1. Draw EF, AG horizontal, meeting CA and BE. The force upon C may be resolved into CF, FE; and the force upon A may be resolved into EF, FA. Hence the vertical forces upon C and A are as CF, AF; and the horizontal forces upon C and A are as FE, EF.

Hence the forces exerted upon C and A are equivalent to two vertical forces whose difference (if F fall without CA) is CA or W; and to a statical couple whose moment is $AC.EF$, or $W.AG$, the moment of the load about the pin A.

COR. 2. The pressure on the pin D is equal to the pressure on the pin C.

COR. 3. The stress on the tie CD is equal to the pressure at C or at D.

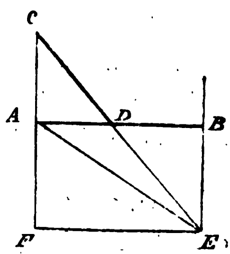
The stress upon a tie in the direction of its length is sometimes called *strain*.

COR. 4. If the beam be horizontal, the total pressures at C and A are respectively

$$\text{on } C = W.\frac{CE}{CA} = W\frac{CE}{CD}.\frac{CD}{CA}$$

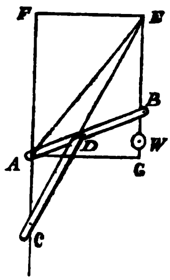
$$= W.\frac{AB}{AD}.\frac{CD}{CA};$$

$$\text{on } A = W.\frac{EA}{CA}.$$



64. ***When a loaded beam is supported by a strut, the points of support being in the same vertical line, to find the magnitude and direction of the stress at the points of support.***

Let AB be a loaded beam, held by a fixt pin at A, and supporting the load W at B; CD the strut, held by a fixt pin at C, and fastened to the beam by a pin at D.



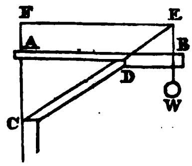
Let CD produced meet, in E, the vertical line through B; join AE.

The load acting in the vertical line EBW is supported by the force at C, which acts in the line CDE, and by a force at A, which must act in the line EA, because the three forces must meet in one point. (Mech. 23.) Hence the three forces which keep the load in equilibrium will be as the lines CE, EA, AC; and therefore, CA representing the load W, CE and EA will represent the forces exerted *by* the points C and A; and EC and AE will represent the forces exerted *upon* the points C and A. Hence the pressures upon C and A are

$$W\frac{EC}{AC} \text{ in direction } EC, \quad \text{and } W\frac{AE}{AC} \text{ in direction } AE.$$

COR. 1. As in the last Proposition, these forces are equivalent to two vertical forces at A (downwards), and at C (upwards), whose *difference* (if F fall without AC) is W; and to a statical couple whose moment is $W.AG$, the moment of the load about the pin A.

COR. 2. The stress upon the strut CD in the direction of its length, is equal to the stress at C or at D.



COR. 3. If the beam be horizontal, the pressure at C is

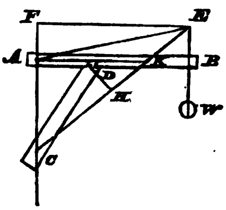
$$W\frac{CE}{CA} = W.\frac{CE}{CD}.\frac{CD}{CA} = W\frac{AB}{AD}.\frac{CD}{CA}.$$

COR. 4. If in this case the end of the strut D rest against a shoulder in the beam AB, to find the pressure on this shoulder in the direction of the beam.

Resolving the pressure in CD in the directions CA, AD, the pressure in the direction AD is

$$\text{pressure on } C\frac{AD}{DC} = W.\frac{AB}{AD}\frac{AD}{AC} = W\frac{AB}{AC}.$$

65. *When a horizontal loaded beam is supported by a mortised strut, the points of support being in the same vertical line: to find the stress at the points of support.*



Let AB be the loaded beam held by a fixt pin at A, and supporting the load W at B; CD the strut mortised at D, and held by a fixt pin at C.

In this case, the frame $ABCD$ acts as a rigid body, and the pressure at C is not necessarily in the direction CD. Let the pressure at C be in the direction CE, meeting in E the vertical line through B. Join EA.

As before, the pressures which support the frame are represented by CE, EA, AC, of which AC represents the load W.

COR. 1. Also as before, the vertical pressures at A and C are as CF upwards, and FA downwards; their difference being equal to the weight W. But, in this case, the pressures are no longer determined by the form of the frame; the whole may be supported at A, or the whole at C, or the

weight may be divided between these two points in any proportion.

The proportion will be determined by the mode of support at A and C. Thus if the pin C rest against a surface, the pressure CE will be perpendicular to the surface.

Cor. 2. The weight supported at C being given, to find the *wrench* upon the mortise.

Let CE be the direction in which the force at C acts: from D (the center of the mortise,) draw DH perpendicular on CE: then the force CE, acting at the distance DH, tends to wrench the strut out of the mortise; therefore the moment of the wrench is as $CE \times DH$.

Let the weight supported at C be nW; therefore

$$CF = n \, . \, CA.$$

$$\text{Also } AK = EF\frac{CA}{CF} = \frac{EF}{n} = \frac{AB}{n};$$

$$DK = AK - AD = \frac{AB}{n} - AD.$$

$$DH = AC\frac{DK}{CK}. \quad \text{Hence}$$

$$\therefore CE \times DH = AC \, . \, DK \, . \, \frac{CE}{CK} = AC \, . \, DK \, . \, \frac{CF}{CA}.$$

Hence the wrench tends to turn DC from DA with a moment

$$= W \, . \, DK \, . \, \frac{CF}{CA} = W\left(\frac{AB}{n} - AD\right)n$$

$$= W(AB - n\,AD).$$

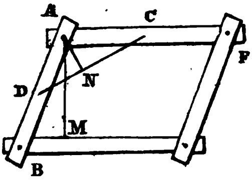
This must be resisted by the cohesion and strength of the materials of the tenon and mortise.

Cor. 1. If the strut at C bear against a vertical surface, CE is horizontal. In this case $n = 0$, wrench $= W \, . \, AB$.

Cor. 2. If n be greater than $\frac{AB}{AD}$, the wrench alters its direction; and the tendency is to turn DC towards DA.

67. *When a frame has an angle stiffened by a brace, and the frame tends to rack by a given force, to find the force exerted by the brace.*

Let FAB be a portion of the frame which tends to rack, by two opposite and equal forces V, W, one acting in AF, and the other at B, parallel to FA. The force W tends to turn the piece AB round A, which tendency is resisted by the brace CD. Hence the moments of the forces about A must be equal, and if P be the force of the brace, and AN be perpendicular to CD,



$$P \times AN = W . AM:$$

that is, the moment of the force of the brace about the angle which it stiffens, is equal to the moment of the racking force.

This is the case whether the brace act as a strut, or as a tie.

Cor. 1. It may hence be proved that if several angles be braced, the sum of the moments of the force of each brace about the angle which it stiffens, is equal to the whole moment of the racking force which acts on the frame.

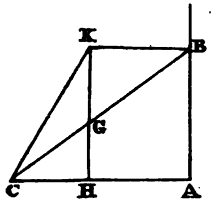
The racking force is always a statical couple.

68. Prop. *A uniform straight beam rests with its extremities against a horizontal and a smooth vertical line: to find the pressures at the extremities.*

We suppose sliding at the lower extremity to be prevented.

Let BC be the beam; AB, AC the horizontal and vertical lines; G the center of gravity. Let HGK be vertical, BK horizontal; join CK. The beam is kept in equilibrium by its weight, which acts in the vertical line KGH; the horizontal pressure at B in the line BK;

and a third pressure at C, which must therefore pass through the point K, and must be in the direction BK. And the triangle KHC has its sides in the directions of these forces, and therefore proportional to them. Hence KH representing the weight of the beam, KC represents the pressure upon C. And this is equivalent to KH, HC; that is, to a vertical force equal to the weight, and a horizontal force HC.



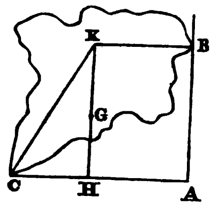
If B be the weight of the beam, H the horizontal force at C, β the angle BCA; we have

$$\frac{B}{H} = \frac{KH}{HC} = \frac{2\,GH}{HC} = 2 \tan \beta.$$

The pressure at B is horizontal, and is equal to H.

Cor. 1. In nearly the same manner may the problem be solved if the beam be not uniform, and G not in the middle of BC.

Cor. 2. And if the body BC be any rigid body whatever, touching the horizontal and vertical planes at B, C, the forces exerted in any position may be determined in the same manner

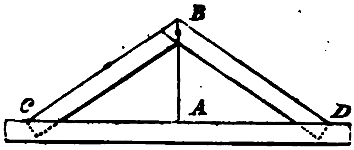


69. Prop. *A simple roof-truss, consisting of equal rafters and a tie-beam, supports given equal loads uniformly distributed along the rafters*: to find the tension of the tie-beam.*

The forces which the rafters exert upon each other at the vertex B, must be horizontal, because the loads are equal.

* The covering of the roof usually does not rest immediately upon the rafters, but upon pieces which lie upon the rafters, parallel to the ridge-tree, and are called *purlines*. Hence the weight is thrown upon special points of the rafter only; but the resultant of all the forces at these points will be a single force.

And the load on each rafter may be supposed to be attached at its middle point. Hence the beam BC is supported as

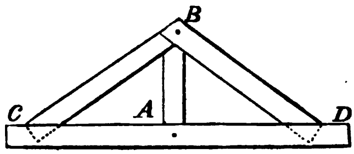


in the last Proposition: and we have, β being the angle BCD, B the weight supported by BC, or BD, and H the horizontal force at C or D, by Art. 68,

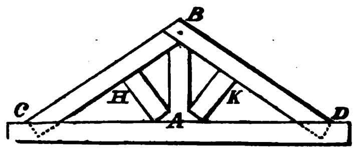
$$\frac{B}{H} = 2\tan\beta; \quad H = \tfrac{1}{2}B\cotan\beta = \tfrac{1}{2}B\frac{AC}{BA}.$$

Cor. The tension is greater as AB is less.

70. The beam CD, which is here considered as a tie only, in practice has weight, and this weight is supported at



C and D. But in practice beams are not of perfectly rigid materials. They are more or less flexible; and thus a beam

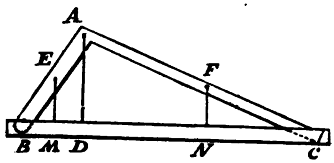


CD, supported at its two ends, will hang down, or *sag*, in the middle. Its strength is by this means diminished. To prevent this result, the middle of the tie-beam CD is supported by a *king-post* BA. Again, the rafters BC, BD will also sag. And to prevent this, these also may have their intermediate points supported by *braces* AH, AK.

The pressure supported by a king-post depends upon the amount of flexure which the material of the tie-beam would undergo by a given force, that is upon the *modulus of elasticity* of the material. This investigation depends upon the consideration of the *strength of materials*. The force exerted by the braces AH, AK, depends in like manner upon the elasticity of the material. These braces may be either struts or ties, according to the conditions of the structure and the materials.

71. *When a scalene roof with a horizontal tie-beam has any weights whatever supported at any points of the rafters: to find the tension of the tie-beam.*

Let R, S be the weights at the points E, F; X the tension; EM, FN, vertical.



Let B, C, be the vertical pressures at B and C. Hence by the lever BC,

$$B.BC = R.CM + S.CN.$$

But the forces B, X, R, acting on the lever AB, whose fulcrum is A, balance: hence taking their moments round A,

$$B.BD - X.AD = R.MD.$$

$$\text{Hence } X = \frac{B.BD - R.MD}{AD} = \frac{B.BC.BD - R.BC.MD}{AD.BC}$$

$$= \frac{(R.CM + S.CN)\,BD - R.BC.MD}{AD.BC}$$

$$= \frac{R\,(CM.BD - BC.MD) + S.CN.BD}{AD.BC}.$$

But

$$CM.BD-BC.MD=BD\,(CD+MD)-(BD+CD)\,MD$$
$$=BD.CD-MD.CD=BM.CD.$$

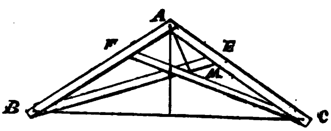
Hence $X=\dfrac{R.BM.CD+S.CN.BD}{AD.BC}$.

The expression is symmetrical, with respect to the two beams, and hence the tension at the other end is the same: as it ought to be.

72. *In a roof with an oblique tie: to find the tension of the tie.*

Let BAC be a roof braced with an oblique tie, BE.

The roof may be considered as a frame urged to rack about A, and preserved by a brace BE. The forces which



urge the frame to rack are the whole weight of the roof acting downwards, and the supporting pressures acting upwards. If the whole weight of the roof be W, and if it be isosceles, a weight $\frac{1}{2}W$ will be supported on each of the walls at B, C; and (B, C being horizontal) the moment of the force to rack the frame is $\frac{1}{2}W.\frac{1}{2}BC$. And by Art. 69, the moment of the force of the brace, BE round A (if this be the only brace,) is equal to the moment just mentioned. Hence drawing AM perpendicular on BE;

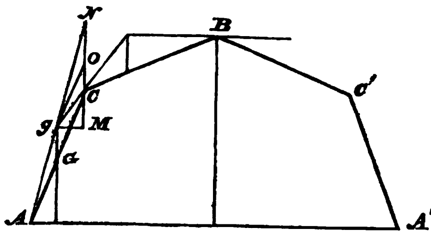
$$\text{force of tie } BE=\tfrac{1}{4}W.\frac{BC}{AM}.$$

Cor. If there be other ties, as CF, the sum of their moments round A will be equal to the moment which tends to rack the roof; and the tension of each tie is not determined by the form of the roof.

73. Prop. *To determine the position of equlibrium of a given isosceles frame, consisting of two pairs of equal beams forming a polygon, and fixt at their lower ends.*

The beams are supposed to be of uniform thickness, so that the centre of gravity of each is in its middle point.

Let AC be one of the lower beams; and G being its center of gravity, let Gg be a vertical line. Then the pres-



sures at A and C must converge to some point in Gg, as g, and their directions will be Ag, Cg. Produce Ag, meeting in N the vertical line drawn through C. Since the beam AC is supported by three pressures in the directions NC, Cg, gN, these forces are as those lines. Hence if NC represents the weight of the beam AC, Ng and gC represent the pressures at A and C. Also drawing gM horizontal, Ng is equivalent to NM, Mg; and gC to gM, MC. Hence Mg represents the horizontal pressure of the beam at A, and gM the equal horizontal pressure at C; NM is the vertical pressure downwards at A, and MC the vertical pressure upwards at C.

Let O bisect CN. Since G bisects AC, g bisects AN, and gO is parallel to AC. Let the angle which CA makes with the horizon be α: then $OgM = \alpha$. And let H be the horizontal pressure at A, A the weight of the beam AC.

$$\frac{\text{pressure upwards at } C}{H} = \frac{MC}{Mg} = \frac{MO}{Mg} - \frac{OC}{Mg} = \tan\alpha - \frac{\frac{1}{2}A}{H}.$$

Hence pressure upwards at $C = H \tan\alpha - \frac{1}{2}A$.

But since the force at B is horizontal, the pressure downwards at C, arising from the beam BC, is the weight of the beam, which call B.

$$\text{Hence } H \tan \alpha - \tfrac{1}{2} A = B.$$

Also we have, by the beam BC, as in Art. 69,

$$\frac{B}{H} = 2 \tan \beta, \text{ or } H \tan \beta - \tfrac{1}{2} B = O.$$

$$\text{Hence } H = \frac{B}{2 \tan \beta}; \quad \frac{B \tan \alpha}{2 \tan \beta} - \tfrac{1}{2} A = B:$$

$$\frac{\tan \alpha}{\tan \beta} = 2 + \frac{A}{B}.$$

This, combined with an equation depending upon the lengths of the beam, will determine α and β.

COR. 1. By the same reasoning,

$$\frac{\text{pressure downwards at } A}{H} = \frac{NM}{Mg} = \frac{MO}{Mg} + \frac{ON}{Mg} = \tan \alpha + \frac{\frac{1}{2} A}{H}:$$

whence pressure downwards at $A = H \tan \alpha + \frac{1}{2} A$.

COR. 2. If there be a polygon consisting of any number of beams, beginning from the highest point, B, B_1, B_2, B_3, and if β, β_1, β_2, β_3 be the angles which these make with the horizon, we shall have, by the same reasoning,

$$B = H \tan \beta_1 - \tfrac{1}{2} B_1,$$
$$H \tan \beta_1 + \tfrac{1}{2} B = H \tan \beta_2 - \tfrac{1}{2} B_2,$$
$$H \tan \beta_2 + \tfrac{1}{2} B_2 = H \tan \beta_3 - \tfrac{1}{2} B_3,$$

Also we have as before $H \tan \beta - \frac{1}{2} B = 0$; and if b, b_1, b_2, b_3 be the lengths of the beams, and h the horizontal distance of A and B, we have, from geometrical considerations,

$$b \cos \beta + b_1 \cos \beta_1 + b \cos \beta_2 + b_3 \cos \beta_3 = h.$$

These five equations determine H, β_1, β_2, β_3, β_4. And a similar process might be followed for any number of beams.

Cor. 3. If at any of the angles, as for example at the lower extremity of B_1, there be a weight C_1, instead of the corresponding equation in the last corollary, we shall have

$$H \tan \beta_1 + \tfrac{1}{2} B_1 + C_1 = H \tan \beta^2 - \tfrac{1}{2} B_2,$$

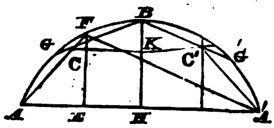
and similarly for any other angle.

74. Prop. *The four beams in last Proposition being of equal lengths and equal weights: to determine the position of equilibrium.*

In this case $A = B$; whence $\dfrac{\tan \alpha}{\tan \beta} = 2 + 1$; hence $\tan \alpha = 3 \tan \beta$.

If the extremities A, A', and the vertex B be given, we may find the figure by the following construction.

The line BH perpendicular to AA' bisects AA'. Bisect AH in E, and erect EF perpendicular to AH, meeting in F the circle passing through ABA'. Join AF, and this will give the position of the lower beam. Take



$BG = AF$, meeting AF in C, and this will give the position of the upper beam: so that BC, CA, will be the positions of the two beams on one side of BH. And a similar figure $BC'A'$ on the other side will be their positions on that side.

For, supposing BC' to meet the circle in G', $A'G' = A'B - BG' = AB - BG = AB - AF = BF$: therefore $A'F$ is parallel to BG', and the angle $BCK = BC'K = FA'E$.

Hence $\tan CAA'$ or $FAE = \dfrac{FE}{AE} = \dfrac{3FE}{A'E}$, because $A'E = 3AE$.

Hence $\tan FAE = 3 \tan FA'E = 3 \tan BCK$, and the condition of equilibrium above found is satisfied.

SECT. II. THE EQUILIBRIUM OF ARCHES—AND FIRST OF DIRECT ARCHES.

75. An arch is a structure composed of heavy masses in contact, supporting each other by the mutual pressure arising from their weight.

In general the masses are several (more than two), and by their juxta-position form, at least approximately, a portion of a vertical ring. This is called the *arch-ring*. The curves which bound the ring below and above are called the *intrados* and the *extrados*. The arch-ring is divided into the separate masses by planes transverse to it; these planes are called the *beds*, *bed-joints*, or *joints*. The separate masses are called *arch-stones*, or *voussoirs*.

The arch-ring rests, at its two extremities, upon supports, which are called *abutments*.

The mass of the arch is supposed to be bounded by vertical planes, each of which is called the *face* of the arch.

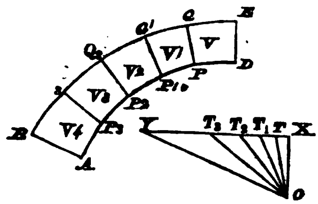
The cylindrical surface, which by its intersection with the face of the arch forms the intrados, is called the *soffit* of the arch.

If the soffit be perpendicular to the face of the arch, the arch is called a *direct* arch; if the soffit be oblique to the face, the arch is called an *oblique* or *skew* arch. In the former case, we have only to consider the relations of form and weight in one vertical plane. In the latter case, it is requisite to consider space of three dimensions.

In the present chapter, we consider the arch as in equilibrium without friction: the effect of friction is taken into account in a subsequent chapter.

76. Prop. *In the equilibrated direct arch, the weight of each voussoir is as the difference of the tangents of the angles which its upper and lower bed-joints make with the vertical.*

Let $ABDE$ be the half of any arch, composed of vous-



soirs, V, V_1, V_2, &c., pressing each other at the bed-joints, PQ, P_1Q_1, &c. Since the arch is equilibrated, the pressure at each of these joints is perpendicular to the joint. And the three forces which act upon each voussoir are the pressures at the upper joint, the pressure at the lower joint, and the weight of the voussoir. And these three forces will be as three lines drawn in their directions, forming a triangle; or (Mech. 17, Cor. 3.) as three lines perpendicular to their directions, forming a triangle.

Draw OX vertical, XY horizontal, and OT, OT_1, OT_2, &c. parallel to the joints PQ, P_1Q_1, P_2Q_2, &c., meeting XY in T, T_1T_2, &c. Then for any voussoir, as V_1, the corresponding triangle OTT_1 has its sides perpendicular respectively to the directions of the three forces which act upon the voussoir; for the pressures at the joints are perpendicular to OT, OT_1, and the weight of the voussoir, being vertical, is perpendicular to the horizontal line TT_1. Hence, OT_1 representing the pressure upon V_1 at the joint P_1Q_1, TT_1 will represent the weight of the voussoir V_1. In like manner, OT_1 representing the pressure upon V_2 at the joint P_1Q_1, T_1T_2 will represent the weight of the voussoir V_2. And the pressures upon V_1 and V_2 at the joint P_1Q_1 are equal, being action and reaction. Hence the weights of the voussoirs V_1, V_2, are represented by TT_1, T_1T_2. And in like manner, on the same scale, the weights of the other voussoirs V_3, V_4, are represented by T_2T_3, T, Y. And so on.

And the lines XT, XT_1, XT_2, XT_3, XY, are the tangents of the angles (OX being the radius) XOT, XOT_1, &c. which the joints make with the vertical OX. Hence XT, TT_1, T_1T_2, T_2T_3, T_3Y, are as the differences of the tangents of these angles; that is, the weights of the voussoirs are as the differences of the tangents of these angles. Q. E. D.

77. Prop. *Given the intrados and the positions of the bed-joints, to find the extrados of the equilibrated arch.*

The same construction being made, and the arch being supposed to be bounded by vertical planes perpendicular to the bed-joints, the weight of each voussoir will be proportional to the area of the voussoir upon their vertical planes. Hence the areas V, V_1, V_2, &c., must be proportional to the lines XT, T_1T_2, T_2T_3, &c.; and the line EQQ_1Q_2, &c., must be drawn so as to satisfy this condition.

If the joints be indefinitely near each other, this condition gives an expression involving differentials, for the curve EQQ_1, &c.: but the solution in this form is of no practical use.

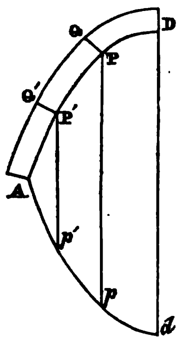
78. Prop. *If the arch-ring be linear and the joints perpendicular to the intrados, the curve of the equilibrated arch will be a catenary.*

The arch-ring is *linear* if the depth of the voussoirs be uniform and very small, the joints being supposed perpendicular to the intrados, that is, to the curve of the arch-ring.

A *catenary* is the curve in which a chain of uniform thickness will hang by the action of gravity, its ends being supported at fixt points.

Since the voussoirs are of small uniform depth, the line of voussoirs may be considered as a heavy line of uniform thickness, and the weight of any portion of this line will be proportional to its length. And since the

joints are perpendicular to the curve, the pressure at each joint will be in the direction of the curve; and the form of the curve will be determined by the condition that the pressures thus exerted at the extremities of any portion are such, in magnitude and direction, as to support that portion.

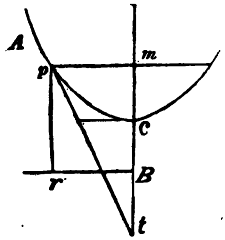


Let Ad be the curve AD inverted with reference to a horizontal line. If a uniform chain Ad hang by the action of gravity, and if vertical lines Pp, $P'p'$, be drawn, the weight of any portion pp' is proportional to the length of the portion pp' or PP'; the tensions at its extremities p, p', are in the direction of the curve; and the form of the curve is subject to the condition that these tensions are such, in magnitude and direction, as to support that portion; for after the equilibrium is established, the flexible part may be supposed to be rigid. (Mech. Int. 26.)

Hence the form of the catenary and of the equilibrated arch are determined by the same conditions; and therefore the total length AD, Ad, and the relative position of the extremities being the same, the form of the curve will be the same in the two cases.

79. Prop. *To find the differential equation to the catenary.*

Let ApC be the catenary, C being the lowest point, at which the curve is horizontal. Let Cm, mp, the vertical and horizontal coordinates of any point p, be x, y; and let the curve Cp be s; the tension at $C = c$. The curve Cp may be conceived to be a rigid heavy body, kept in equilibrium by its weight, and by



the tensions at C and p. And these three forces are in the directions mC, pm, and the tangent at p. Let pt be the tangent at p, meeting mC in t. Then the three forces are in the direction of the three sides of the triangle pmt, and are as those sides.

$$\text{Hence } \frac{\text{tension at } C}{\text{weight of } Cp} = \frac{mp}{mt}.$$

$$\text{But } \frac{mp}{mt} = \frac{dy}{dx},$$

$$\text{hence } \frac{c}{s} = \frac{dy}{dx};$$

which is the differential equation to the catenary.

80. PROP. *The tension at any point of the catenary is equal to the tension at the lowest point, plus the weight of a length of the chain equal to the height above the lowest point.*

By the above reasoning we have

$$\text{tension at } p = \text{weight of } Cp \cdot \frac{pt}{mt}.$$

$$\text{But } \frac{pt}{mt} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left(1 + \frac{c^2}{s^2}\right)} = \frac{\sqrt{(c^2+s^2)}}{s},$$

Hence if s be the weight of Cp,

$$\text{tension at } C = s\frac{\sqrt{(c^2+s^2)}}{s} = \sqrt{(c^2+s^2)}.$$

$$\text{But since } \frac{c}{s} = \frac{dy}{dx},$$

$$\frac{\sqrt{(c^2+s^2)}}{s} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \frac{ds}{dx}.$$

$$\text{Hence } \frac{s}{\sqrt{(c^2+s^2)}} \frac{ds}{dx} = 1;$$

and integrating, $\sqrt{(c^2+s^2)} = x + C$; the quantity c is added, so that when $x = 0$, $s = 0$.

$$\text{Hence tension at } p = x + c,$$

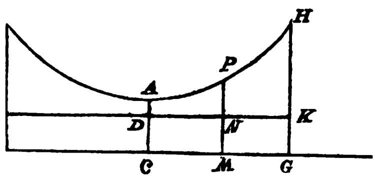
where c is the tension at the lowest point C, and x the weight of a length of chain Cm.

Cor. 1. If CB be taken vertically downwards from $C = c$, and Br, a horizontal line, be drawn, and pr vertical, the tension at any point p is equal to the weight of a length of chain pr. For since $CB = c$, $pr = x + c$.

Cor. 2. Hence in the equilibrated linear arch, the pressure at any point is equal to the pressure at the vertex of the arch, *plus* the weight of a length of the arch-ring, equal to the depth of the point below the vertex.

81. Prob. *A flexible line, suspended from fixt points, is loaded with weights, at each point, proportional to the vertical ordinate and to the horizontal length; to find the form of the curve.*

The weights are supposed to be uniformly distributed along the horizontal abscissa, x. Hence, the weights being



represented by the ordinate y, the sum of all the quantities $y\delta x$ (δx being a small increment of x) is the whole weight; that is, $\int_x y$. And this, taken from the origin to a point P, is the whole weight of the part ending at P.

If a be the horizontal tension at the lowest point A, the weight on AP is supported by two forces, at A and at P; and we shall have, by the triangle of forces,

$$\text{weight on } AP = a\frac{dy}{dx}; \text{ whence } \int_x y = a\frac{dy}{dx};$$

$$\text{therefore } y = a\frac{d^2y}{dx^2}; \quad y\frac{dy}{dx} = a\frac{dy}{dx} \cdot \frac{d^2y}{dx^2};$$

$$y^2 - b^2 = a\left(\frac{dy}{dx}\right)^2; \; b \text{ being the ordinate at } A = AC.$$

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{y^2 - b^2}}\frac{dy}{dx}; \quad \frac{x}{\sqrt{a}} = \mathrm{l}\,\frac{y + \sqrt{y^2 - b^2}}{b}.$$

$$\text{Hence } y = \frac{b}{2}\left(e^{\frac{x}{\sqrt{a}}} + e^{-\frac{x}{\sqrt{a}}}\right).$$

Cor. 1. This result is approximately applicable to a suspension bridge; the vertical suspending rods and the road being supposed to be the principal part of the weight, and the rods being supposed to be uniformly distributed along the horizontal line.

Cor. 2. If in this case, in any ordinate MP we take MN to NP as the weight of any portion of the road to the weight of the corresponding portion of the rods, the horizontal line DN will be the roadway.

Cor. 3. Let AD be given $= c$, and let m to n be the ratio of weights of corresponding portions of the road and of the rods, at the lowest point A. Then

$$AC = b = \frac{m+n}{n}c, \text{ is known.}$$

Also any other point of the curve being given, a is known by the equation

$$\frac{x}{\sqrt{a}} = \mathrm{l}\,\frac{y + \sqrt{y^2 - b^2}}{b};$$

and hence the curve may be constructed.

COR. 4. Let the whole horizontal length be given, ($= 2DK$, if the suspension curve be symmetrical,) and the height of the point of suspension above the roadway, ($=HK$). Let $DK = k$, $KH = h$. Also, in the immediate neighbourhood of the points of suspension, let the weights of corresponding portions of the road and of the suspension rods (for a unit of the road) be r and s.

$$\text{Then } GK = \frac{r}{s}h, \quad GH = \frac{r+s}{s}h.$$

$$\text{Hence } \frac{r+s}{s}h = \frac{b}{2}\left(e^{\frac{k}{\sqrt{a}}} + e^{-\frac{k}{\sqrt{a}}}\right).$$

COR. 5. Also

$$\text{weight of } y = a\frac{dy}{dx} = \frac{b\sqrt{a}}{2}\left\{e^{\frac{x}{\sqrt{a}}} - e^{-\frac{x}{\sqrt{a}}}\right\}.$$

$$\text{Hence, } r + s = \frac{b\sqrt{a}}{2}\left\{e^{\frac{k}{\sqrt{a}}} - e^{-\frac{k}{\sqrt{a}}}\right\},$$

COR. 6. Hence r, s, and h, k, being given, b and a are determined by the two equations in Cor. 4 and 5, and the curve may be constructed.

Here a is a quantity of two dimensions; namely, a unit of the linear measure of the road, multiplied into such a length of the suspending rod (taken for a unit of the road) as will measure the tension at A by its weight. And r, s, are in like manner quantities of two dimensions.

SECT. III. THE EQUILIBRIUM OF OBLIQUE ARCHES.

82. THE total mass of an arch, direct or oblique, is bounded by a surface above (generally level) which is termed the *roadway*; a surface below, generally cylindrical, which is termed the *soffit*; and two upright surfaces limiting the

breadth of the roadway and of the structure, which are termed the *faces*. In the direction of the roadway, the arch is bounded at each end by the *abutments*.

The soffit may be supposed to be *any cylindrical* surface, that is, a surface generated by a line moving parallel to a fixt horizontal *axis*, and guided by any curve; which curve would form the *transverse section* of the cylindrical surface.

The soffit meets the face in a line termed the *intrados*. The structure, where it is bounded by the intrados, consists of masses termed *voussoirs*, or *arch-stones*. The surfaces in which these touch each other are called *beds*, or *bed-joints*. The bed-joints are continued along the soffit, and divide it into *courses*. These courses are subdivided into separate stones (or bricks) by other surfaces transverse to their length, which surfaces are *joints*, or *cross-joints*.

The abutments are always the masses which resist the thrust in the direction of the roadway, at each end of the bridge, and thus support the structure.

If the horizontal axis of the soffit be perpendicular to the direction of the roadway, as in a direct arch, the courses on the soffit will all be parallel to the axis, the pressures on the beds will be in planes perpendicular to the axis, and therefore in vertical planes. The lines of these pressures will fall within the structure so far as its vertical faces are concerned; and hence all the pressures may be resisted by abutments at the end of the arch: for these, being placed in the length of the roadway, are in the planes in which the pressures are.

But if the horizontal axis of the soffit be not perpendicular to the roadway, and if the courses be horizontal, the pressures on the beds will still be in vertical planes

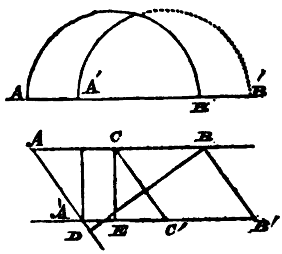
perpendicular to the axis, and therefore oblique to the roadway. Therefore the lines of these pressures will not all fall within the structure, and the pressures cannot all be resisted by the abutments. Hence in such an arch, namely in an oblique arch, the courses must run in some other manner in order that the equilibrium of the structure may be maintained.

83. A *direct arch* (as before stated) is one in which the axis of the soffit is perpendicular to the roadway: an oblique arch is one in which the axis is not perpendicular to the roadway. The angle which the axis makes with a perpendicular to the roadway is the *obliquity* of the arch.

For the sake of distinctness we shall suppose the arch to stand over a *canal* which is bounded by lines drawn parallel to the axis through the extremities of the intrados.

The *span* of the arch is the horizontal distance of the extremities of the intrados.

Let the figure represent the elevation and plan of an arch; AB being the direction of the roadway, CC' the direction of the axis and of the canal; $A'B'$ the other side of the roadway, and AA', BB' the sides of the canal. Draw BD perpendicular on AA', and CE perpendicular on $A'B'$. Then CE is the breadth of the roadway, and BD the breadth of the canal; CC' is the length of the axis; AB is the span of the arch; the angle $C'CE$ or ABD is the obliquity. It appears from the figure that



(since $BD = AB \times \cos ABD$, $CE = CC' \times \cos C'CE$;)

the breadth of the canal is equal to the span multiplied into cosine of obliquity; and the breadth of the roadway

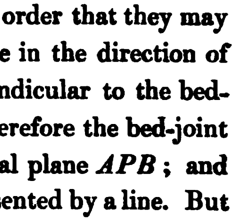
is equal to the length of the axis multiplied into cosine of obliquity.

84. When the bed-joints are of such a form that the arch is in equilibrium without friction, the courses are called the *equilibrated courses*.

When the bed-joints make a constant angle with the line which generates the cylindrical surface of the soffit, the courses are called *spiral courses;* this case will be considered in another Chapter.

85. PROB. *To determine the form of the equilibrated courses when all the bed-joints on the face of the oblique arch pass through the same center.*

Let a portion of the structure be taken, bounded by two vertical planes AB, $A'B'$, and let the whole figure be a projection of the arch on a vertical plane, so that APB, $A'P'B'$ are the sections of the soffit by the two vertical planes. In the case of equilibrium, the pressures at the bed-joints, as at P, must be in a vertical plane APB, in order that they may be resisted by the abutments, which are in the direction of the line AB. But the pressure is perpendicular to the bed-joint, because friction is neglected. Therefore the bed-joint at P must be perpendicular to the vertical plane APB; and therefore, in the projection, will be represented by a line. But all the joints in the face APB pass through one center C. Therefore PC represents the direction of the bed-joint at P, and if we draw a semicircle APB with center C, the bed-joint will cut this semicircle at right angles at P.



In like manner, if $A'P'B'$ be another vertical section, the bed-joints being still governed by the same rule, will be

in the direction $P'C'$; and the bed-joints will cut the semicircle $A'P'B'$ at right angles at P'. And so on for other sections.

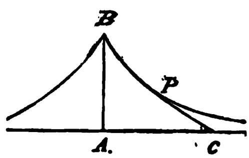
Hence if we draw an indefinite number of semicircles APB, $A'P'B'$, &c., the curve which in our projection represents the bed-joints passing through P, will cut all these semicircles at right angles.

Therefore this curve has a constant tangent. For the lines PC, $P'C'$, are manifestly tangents to this curve at the points P, P': and these are all equal to AC.

Hence, *the equilibrated courses are determined by this condition, that the projections of the bed-joints on a vertical plane are curves having a constant tangent.*

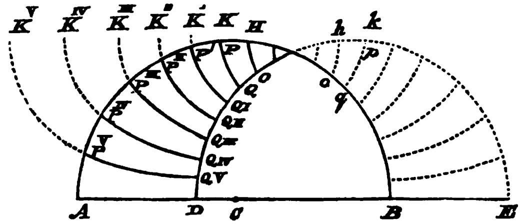
86. The curve of which the tangent intercepted by the line of abscissas is constant, is sometimes called the *Tractrix*, or *Tractory*; as being the curve which a point (P) would describe when drawn by a string of given length (PC), of which the other end moves along a straight line, (AC); it being supposed that the body P moves upon a horizontal plane APC, on which the friction is such as to make it stop when T stops. In this case, the body P will at every point move in the direction PC, and PC will be a tangent. Now PC is constant. Hence the curve thus described is the one to which we are led by the solution of our Problem.

This curve has a cusp at B, where PC becomes perpendicular to the abscissa; the abscissa is an asymptote to it, and it has another similar branch on the other side of the cusp, which also has the abscissa for an asymptote.



87. Hence the general form of the courses is evident. The projections of the bed-joints on the vertical plane will be a series of tractories, PQ, $P'Q'$, $P''Q''$, $P'''Q'''$, and so on; of

which the cusps K, K', K'', &c. are all in the horizontal line

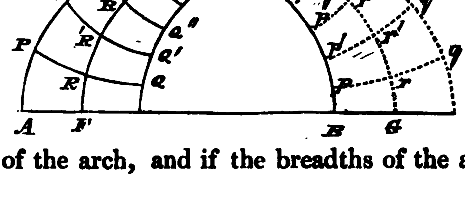


drawn through H, the summit of the semicircle APB. The bed-joint passing through H will be a portion of the tractory beginning at the cusp. In like manner, the bed-joint passing through to the summit of the semicircle on the other face, will be a portion oh of the other branch of a tractory of which the cusp is at h. And the courses on the other side of the summit will all be portions of the other branches of tractories, as qp, which belongs to a tractory having its cusp at k.

88. The above conclusions are applicable to the case where the intrados on the face is a semicircle, and the bed-joints perpendicular to the intrados. [The point A or B is the *spring* of the arch; and of these, A is the *obtuse spring*, and B the *acute spring;* these denominations being taken from the angles on the ground-plan.]

Prop. *If the equilibrated courses make similar sections on the two faces of the arch, the breadths of the arch-stones on each face will go on diminishing from the obtuse spring to the acute spring.*

If a vertical plane FG be drawn midway between the



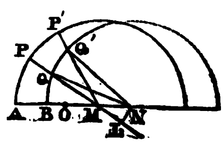
two faces of the arch, and if the breadths of the arch-stones

on the section of this plane with the soffit be all equal, the sections of the courses on the two faces will be similar, because this plane is similarly related to the two faces. Let R, R', R'' be the points where the bed-joints intersect this vertical plane, so that FR, RR', $R'R''$, &c., are all equal; and let tractories PQ, $P'Q'$, $P''Q''$, &c. be drawn through the points R, R', R''. Then it is plain by the form of the tractory, that the breadth PP'' is greater, and QQ' less, than RR'; $P'P''$ is greater, and $Q'Q''$ less, than $R'R''$; and so on. Also, it is evident that these pairs of breadths approach more and more nearly to equality as we approach to the summit G of the midway curve FG. At this point the breadth of the course on the two faces is equal. Beyond this point, $p'p''$ becomes less than $r'r''$, and $r'r''$ less than $q'q''$; and the inequality goes on augmenting all the way to the acute spring B.

Hence on the side of A the breadths PP', $P'P''$, &c. are all greater than RR'; on the side of B, the breadths $p'p''$, pp', &c. are all less than rr'; that is, than RR'; and this excess goes on increasing from A, till it becomes equality at G, and defect from G to B; that is, the breadth of the arch-stones goes on diminishing from A to B.

89. PROB. *Given the number of courses (similar on the two faces,) to find the breadth of each course along the semicircle on the face.*

Let AP, BQ be the projections of the semicircles drawn in each of two vertical planes, parallel to the face, and very near each other. Let P, Q be two points in the projected bed-course passing through P; M, N the centers of the semicircles; then PM, QN are tangents to the curve. Let NL be perpendicular on PM produced, and let the radius AM, BN, PM or $QN = a$; $MN = h$. The angle $AMP = \theta$.



We have $NL = MN . \sin\theta = h \sin\theta$; and angle $MQN = \frac{NL}{NQ}$, nearly, because the angle is very small.

$$\text{Hence angle } MQN = \frac{h \sin \theta}{a}.$$

$$\text{Hence } BNQ = AMQ - MQN = \theta - \frac{h \sin \theta}{a}.$$

$$\text{In like manner, if } AMP' = \theta', \; BNQ' = \theta' - \frac{h \sin \theta'}{a}.$$

$$\text{Hence} \quad QNQ' = BNQ' - BNQ = \theta' - \theta - \frac{h}{a} (\sin \theta' - \sin \theta),$$

$$\text{whence } PMP' - QNQ' = \frac{h}{a} (\sin \theta' - \sin \theta).$$

$$\text{Let } PMP' = \theta' - \theta = \phi, \text{ and } QNQ' = \phi - \delta.$$

$$\text{Hence } \delta = \frac{h}{a} \{\sin (\theta + \phi) - \sin \theta\}.$$

If $OM = x$, h is the increment of x, when MN is very small, and δ the corresponding decrement of ϕ. Hence at the limit

$$\frac{d\phi}{dx} = -\frac{\delta}{h} = -\frac{1}{a} \{\sin (\theta + \phi) - \sin \theta\},$$

$$\frac{1}{\sin (\theta + \phi) - \sin \theta} \frac{d\phi}{dx} = -\frac{1}{a}.$$

Instead of integrating this, we shall simplify it by supposing ϕ small;

$$\sin (\theta + \phi) - \sin \theta = 2 \cos \left(\theta + \frac{1}{2}\phi\right) \sin \frac{1}{2}\phi$$

$$= \phi \cos \theta, \text{ nearly, when } \phi \text{ is small.}$$

$$\text{Hence } \frac{1}{\phi} \frac{d\phi}{dx} = -\frac{\cos \theta}{a}; \quad l \frac{\phi}{\phi_0} = -\frac{x \cos \theta}{a};$$

ϕ_0 being the value of ϕ when $x = 0$.

If the distance AB, for the whole arch, be c,

$$\mathrm{l}\,\frac{\phi}{\phi_0} = -\frac{c\cos\theta}{a}\quad \phi = \phi_0\, e^{-\frac{c\cos\theta}{a}};$$

and if b be the length of the axis, β the obliquity,

$$c = b\tan\beta;\quad \phi = \phi_0\, e^{-\frac{b}{a}\tan\beta\cos\theta}.$$

90. Let $x = 0$ for the ***mid-way section*** of the arch.

If the number of courses in the whole of the mid-way semicircle be n, $\phi_0 = \dfrac{\pi}{n}$;

$$\text{whence } \phi = \frac{\pi}{n}\, e^{-\frac{b}{a}\tan\beta\cos\theta}.$$

This applies to the second face, in which x is positive. For the first face we have

$$\phi = \frac{\pi}{n}\, e^{\frac{b}{a}\tan\beta\cos\theta}.$$

For the lowest course, make $\theta = 0$; hence

$$\phi_1 = \frac{\pi}{n}\, e^{\frac{b}{a}\tan\beta}.$$

Put this value for θ; and we have for the second course,

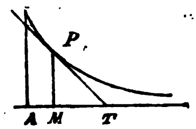
$$\phi_2 = \frac{\pi}{n}\, e^{\frac{b}{a}\tan\beta\cos\phi_1};\ \text{similarly,}$$

$$\phi_3 = \frac{\pi}{n}\, e^{\frac{b}{a}\tan\beta\cos\phi_2};\ \text{and so on.}$$

When we go beyond the summit, $\cos\theta$ becomes negative, and ϕ is less than $\dfrac{\pi}{n}$.

91. *To find the equation to the tractory.*

The tractory is the curve in which the tangent intercepted by the line of abscissas is constant; as stated Art. 86.



Let AM, MP, the abscissa and ordinate, be x and y; PT the tangent $= a$. Then

$$\sqrt{\left\{y^2 + y^2\left(\frac{dx}{dy}\right)^2\right\}} = a,$$

$$\text{whence } -\frac{\sqrt{(a^2-y^2)}}{y}\cdot\frac{dy}{dx} = 1;$$

$$\text{or } -\frac{a^2}{y\sqrt{(a^2-y^2)}}\frac{dy}{dx} + \frac{y}{\sqrt{(a^2-y^2)}}\frac{dy}{dx} = 1.$$

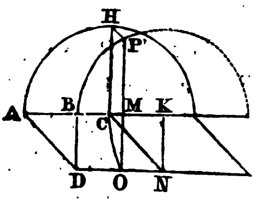
Hence (Doctrine of Limits, VI. 18).

$$a\,\mathrm{l}\,\frac{a+\sqrt{(a^2-y^2)}}{y} - \sqrt{(a^2-y^2)} = x,$$

which makes x vanish when $y = a$. Hence x is measured from A, where the curve is perpendicular to the abscissa.

92. PROB. *In the case of Art.* 88, *To find the equation to the equilibrated bed-joints on the horizontal plane.*

Let the soffit be continued till the bed-joint passes through the summit, and let CAH be the vertical plane passing through this point.



Let HP be the projection of the bed-joint on the vertical plane; BP the semicircle passing through P, BD perpendicular to AB, meeting in D the line AD parallel to the axis of the arch. The point B is the projection of the point D: hence BP is the projection of the semicircle which stands upon DN, a line parallel to AC. And if PMO be drawn perpendicular

to AB, meeting DN, O is a point in the horizontal projection of the same bed-joint. Let $CM = x$, $MP = y$, $CN = u$, $NO = v$, NK perpendicular to AC, $CNK = \beta$. Then $v = NO = MK = \sqrt{(a^2 - y^2)}$, for K is the center of the semicircle BP.

Hence, by Art. 91, $a \, \mathrm{l} \dfrac{a + v}{\sqrt{a^2 - v^2}} - v = x$.

But $NO = MK = CK - CM$; or $v = u \sin \beta - x$.

Hence $u \sin \beta = x + v = a \, \mathrm{l} \dfrac{a + v}{\sqrt{a^2 - v^2}} = \dfrac{a}{2} \mathrm{l} \dfrac{a + v}{a - v}$.

Whence the equation to the curve CO (to co-ordinates CN, NO) is

$$u \sin \beta = \frac{a}{2} \mathrm{l} \frac{a + v}{a - v}.$$

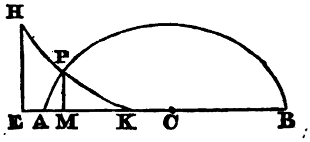
93. We have hitherto supposed the intrados on the face of the arch to be a semicircle with the bed-joints all perpendicular to the curve; or at least, to have the bed-joints all converging to one center. But if we suppose the bed-joints to have any other position;—if for instance, the *intrados on the face* be an *ellipse*, and the joints perpendicular to the curve, we may find the projections of the bed-joints in nearly the same manner as before.

Let APB be the vertical plane curve which is perpendicular to all the bed-joints; and let $A'P'B'$ be the projection of another of these vertical planes upon APB. Then the curve PP', which is the projection of the bed-joints, is every where perpendicular to the curves $A'P'$; and may be determined by that condition.

It is easy to see that the general form of the equilibrated courses, and their relative position, will be similar in this case and in the one before considered.

94. Prob. *When the intrados on the face of the arch is an ellipse (the bed-joints being perpendicular to the intrados): to find the vertical projections of the courses.*

Let $EM = x$, $MP = y$, the abscissa and ordinate of the projection HP; C the center of the ellipse; $CM = u$; a, b, the semi-axes; PK a normal to the ellipse.



$$\text{Then, } MK = \frac{b^2}{a^2}u = \frac{b^2}{a^2}\cdot\frac{a}{b}\sqrt{(b^2 - y^2)} = \frac{b}{a}\sqrt{(b^2 - y^2)},$$

$$\text{and } -\frac{dy}{dx} = \frac{MP}{MK} = \frac{ay}{b\sqrt{(b^2 - y^2)}};$$

$$\text{hence } -\frac{\sqrt{(b^2 - y^2)}}{y}\cdot\frac{dy}{dx} = \frac{a}{b}.$$

This may be integrated nearly as before, Art. 91;

$$\text{and we have } -\frac{b^2}{\sqrt{b^2 - y^2}}\frac{dy}{dx} + \frac{y}{\sqrt{(b^2 - y^2)}}\frac{dy}{dx} = \frac{a}{b};$$

$$b\,\mathrm{l}\,\frac{b + \sqrt{b^2 - y^2}}{y} - \sqrt{(b^2 - y^2)} = \frac{a}{b}x;$$

and when $y = b$, $x = 0$;

$$\frac{b^2}{a}\,\mathrm{l}\,\frac{b + \sqrt{b^2 - y^2}}{y} = x + \frac{b}{a}\sqrt{(b^2 - y^2)}.$$

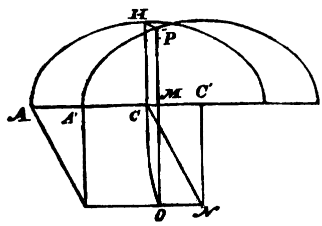
95. Prob. *In this case, to find the horizontal projection of the courses.*

As before, in Article 92, let

$CN = u$, $NO = v$, $CM = x$,

$MP = y$;

$v = NO = MC' = \frac{a}{b}\sqrt{(b^2 - y^2)}$;



$$\sqrt{b^2 - y^2} = \frac{bv}{a}; \; y = \frac{b}{a}\sqrt{a^2 - v^2},$$

$$\therefore x = CM = CC' - NO = u \sin\beta - v.$$

Hence the equation of Art. 94 becomes

$$\frac{b^2}{a} \operatorname{l} \frac{b + \dfrac{bv}{a}}{\dfrac{b}{a}\sqrt{a^2 - v^2}} = u \sin\beta - v + \frac{b^2 v}{a^2};$$

whence

$$u \sin\beta = \frac{b^2}{2a} \operatorname{l} \frac{a + v}{a - v} + \frac{a^2 - b^2}{a^2} v.$$

CHAPTER V.

THE EQUILIBRIUM OF MACHINES AND STRUCTURES, FRICTION BEING TAKEN INTO THE ACCOUNT.

SECTION I. OF FRICTION.

96. It has been stated, (Mech. Int. 35.) that bodies have not, in fact, perfectly smooth surfaces, as in the preceding Sections they are supposed to have. They oppose a resistance to bodies sliding along their surface; and this resistance modifies the motion which takes place; and the same resistance may prevent the motion taking place altogether. In both cases it is called *Friction**. In the present Section we consider only its *statical* effect, that is, its effect in preventing motion.

97. If a heavy body resting with its plane surface upon a horizontal plane, be drawn along by a horizontal force, the force which is requisite to move the body, is the amount of the Friction. If there were no friction, the plane and the body being perfectly smooth, the smallest possible force would put the body in motion; the degree of velocity communicated being, however, dependent upon the amount of the force, and upon the the mass of the body. But friction acting, the force which is just requisite to put the body in motion must overcome the friction, and therefore exactly measures the friction. Or rather, what is requisite in order that the force may measure the friction, is, not that the force should overcome the friction and put the body in motion,

* In the case in which motion is prevented, the force which is here called Friction has sometimes, in familiar language, been termed *Stiction*.

but only that it should balance the friction, and thus put the body in a state of indifference to rest or motion: or, as it is sometimes expressed, should put the body in *a state bordering on motion.*

In the same manner in any machine, if the parts be perfectly smooth, there is an exact proportion among the forces, which must obtain in order that the machine may be in equilibrium: and on this hypothesis, if the forces have not this proportion, motion will ensue. But in fact, friction will, up to a certain limit, prevent this motion from taking place. Not an indefinitely small increase, but a finite increase of any of the forces will be requisite to balance the friction, and to put the machine in a state bordering upon motion.

98. The force of Friction which thus prevents motion when it would otherwise ensue, is a *passive* force. The operation is to prevent, but not to produce motion. The force is called into play by having forces to resist; but if not so produced, the force does not exist. When a body is pulled sideways upon a horizontal plane, by a force which gradually increases, the friction increases till it reaches its limit. If the whole friction be a force of 10 pounds, the force which pulls the body may become 1, 2, 3, 4, 5 pounds and so on, and the quantity of friction called into play in each case, will be exactly equal to these forces. When the force becomes 10 pounds, the body is put in a state bordering upon motion; when the force becomes 10 pounds and a fraction (however small), the body is put in motion.

But as we have said, the above forces of 1, 2, 3, 4, 5 pounds can act only to prevent, not to produce motion. If the extraneous force P be 3 pounds acting in the direction AB, the friction F will be 3 pounds acting in the direction

BA. But if the force *P* do not act, there is no force *F* acting in the direction *BA*, either to produce pressure or motion.

99. Also the force of friction changes its direction, according to the direction of the force which it has to resist. If the force *P* act in the direction *AB*, the friction *F* acts in the direction *BA*. But if, the same body still remaining at rest under the same conditions, the force *P* acts in another direction *AC*, the friction *F* changes its direction to *CA*. And if the force *P* act in the direction *BA*, opposite to that first supposed, the friction *F* will act in the direction *AB*.

100. If we suppose a body acting upon an horizontal plane, capable of moving only in two opposite directions *AB* and *BA*, and of such a weight and kind that its total friction is ten pounds; this body will remain at rest when acted upon by any force less than ten pounds, either in the direction *AB* or *BA*. The equilibrium will subsist *between the limits* $P = \pm 10$, according to our usual mode of representing forces.

In like manner, in any machine or structure, friction being supposed to operate, the equilibrium will subsist between certain limits. And it is only when the forces transgress these limits on one side or the other that the equilibrium is destroyed. In the present Chapter we investigate these limits for various machines and structures, according to the known laws of friction.

101. The laws of Friction are to be determined by means of experiments; and these determinations have been made with care, especially in modern times*. The general result of the experiments may be stated as follows.

* Nouvelles Expériences sur le Frottement, faites à Metz, en 1831, 1832, 1833, imprimées par ordre de l'Acad. des Sc. 1834.

The Friction of sliding is independent of the surface in contact, proportional to the pressure, and independent of the velocity of motion.

The Friction bears to the pressure upon the surface a ratio which is constant for the same substances with their surfaces in the same condition, but various when the materials and the state of the surfaces vary. If P be the pressure on the surface, the friction will be fP, f being a constant coefficient.

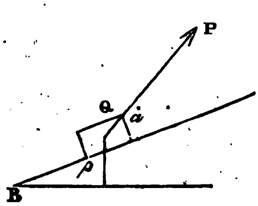
102. Values of f according to the experiments of Morin.

Soft Limestone on Soft Limestone	.64
Iron on Oak	.62
Cast Iron on Oak	.49
Oak on Oak, (fibres parallel)	.48
Leather Belts on Wooden Pullies	.47
Hard Limestone on Hard Limestone	.38
Leather Belts on Cast Iron Pullies	.28
Brass on Brass	.20
Wrought Iron on Cast Iron	.19
Cast Iron on Elm	.19
Brass on Iron	.16
Cast Iron on Cast Iron	.15
Wrought Iron on Wrought Iron	.14
Oak on Oak greased	.10
Cast Iron on Cast Iron greased.............	.10
Brass on Iron greased	.08
Pivots of Wrought or Cast Iron on Brass or Cast Iron pillows:	
—— continually supplied with oil	.05
—— greased from time to time	.08
—— dry	.15

SECTION II. THE EQUILIBRIUM OF MACHINES WITH FRICTION.

103. Prop. *A force acting in any direction supports a weight upon an inclined plane: to determine the conditions of equilibrium, friction being taken into the account.*

Let P be the force, acting at an angle a with the plane; Q the weight, β the angle of inclination of the plane to the horizon.



The force P which acts on Q may be resolved into $P \cos a$, parallel to the plane, and $P \sin a$, perpendicular to the plane. So the force Q may be resolved into $Q \sin \beta$ acting down the plane, and $Q \cos \beta$, acting perpendicular to the plane. Hence the force acting up the plane is $P \cos a - Q \sin \beta$, and the force acting perpendicularly towards the plane is $Q \cos \beta - P \sin a$. Hence the friction along the plane is (Art. 101;)

$$f(Q \cos \beta - P \sin a).$$

And in order that the weight Q may be on the point of ascending, the force up the plane must be equal to the friction, that is

$$f(Q \cos \beta - P \sin a) = P \cos a - Q \sin \beta; \text{ whence}$$

$$P = Q \frac{\sin \beta + f \cos \beta}{\cos a + f \sin a}.$$

In like manner, in order that the weight Q may be on the point of descending, we must have

$$f(Q \cos \beta - P \sin a) = Q \sin \beta - P \cos a; \text{ whence}$$

$$P = Q \frac{\sin \beta - f \cos \beta}{\cos a - f \sin a}.$$

These formulæ give the limits of P.

Cor. 1. If P acts along the plane, $a = 0$,

$$P = Q(\sin\beta \pm f\cos\beta).$$

Cor. 2. When the body is on the point of descending,

$$P = Q(\sin\beta - f\cos\beta),$$

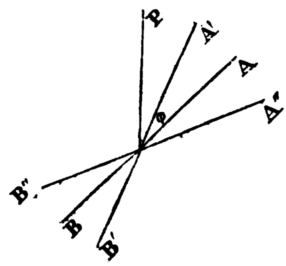
and the body will be supported, if P be not less than this. Hence if $\sin\beta - f\cos\beta$ be 0 or negative, the body will be supported, even if P be 0; that is, the body will not descend though unsupported, if $\sin\beta < f\cos\beta$, or if $\tan\beta < f$.

Cor. 3. Let $f = \tan\phi$; then the body unsupported will not slide on the inclined plane if $\tan\beta < \tan\phi$; or if $\beta < \phi$. The angle ϕ is called the *angle of sliding*.

Cor. 4. Putting $\tan\phi$ for f in the formula when Q is on the point of ascending, we have

$$P = Q\frac{\sin\beta + \tan\phi\cos\beta}{\cos a + \tan\phi\sin a}$$

$$= Q\frac{\sin(\beta + \phi)}{\cos(a - \phi)},$$



which is the formula that we should have, if instead of a plane AB, we had a plane $A'B'$ making an angle ϕ below AB.

Cor. 5. In like manner for the case when Q is on the point of descending, we have

$$P = Q\frac{\sin(\beta - \phi)}{\cos(a - \phi)},$$

which is the formulæ that we should have for a plane $A''B''$ making an angle ϕ with AB.

Cor. 6. In the case in which the force P acts horizontally, $a = -\beta$.

Hence, for the limit at descent, we have

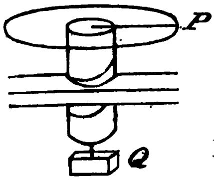
$$P = Q \frac{\sin(\beta - \phi)}{\cos(\phi - \beta)} = Q \tan(\beta - \phi);$$

for the limit at ascent,

$$P = Q \frac{\sin(\beta + \phi)}{\cos(-\beta - \phi)} = Q \tan(\beta + \phi).$$

104. Prop. *When forces balance each other on the screw, to find the conditions of equilibrium, friction being taken into the account.*

Let a force P acting at an arm a, keep in equilibrium a force Q, acting according to the length of the screw; the radius of the cylinder on which the thread of the screw is described being b, and the angle which the thread makes with the plane perpendicular to the axis being β.



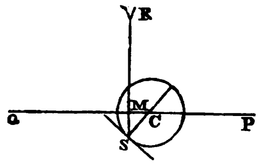
We may suppose the whole of the weight Q to be supported at a single point of the thread of the screw, by a force acting horizontally: which force will be $\frac{Pa}{b}$. Hence by Art. 103, Cor. 6.

$$\frac{Pa}{b} = Q \tan(\beta \pm \phi) = Q \frac{\tan\beta \pm f}{1 \mp f \tan\beta}.$$

This mode of considering the problem supposes the force P to be symmetrically distributed about the axis; for if it were not, the force P would produce a pressure against the side of the screw, which would occasion friction.

105. Prop. *When any lever is acted upon by any forces: to find the conditions of equilibrium, taking into account the friction of the axle on its support.*

Let the resultant of the forces which act upon the lever be R, acting in the direction RS; therefore RS must pass through the point of contact of the axle with its support. Let S be the point of support, SC perpendicular to the surfaces there in contact. Then the angle RSC must be not greater than the angle of sliding, in order that there may be equilibrium. Hence the limit of equilibrium is determined by this condition, $\tan CSR = f$. And the same condition, CS being on the other side of RS, determines the other limit.



Cor. 1. Hence $\sin CSM = \dfrac{f}{\sqrt{1+f^2}}$.

Cor. 2. If the forces P, Q, which act upon the lever, be parallel, $R = P + Q$: and if $CP = a$, $CQ = b$, $CS = r$, the limits of equilibrium are determined by the equation

$$Pa = Qb \pm \frac{fr\,(P+Q)}{\sqrt{(1+f^2)}}.$$

For, RS meeting PQ in M,

$$CM = CS \sin CSM = \frac{fr}{\sqrt{(1+f^2)}}.$$

And $P.PM = Q.QM$; that is

$$P\left(a + \frac{fr}{\sqrt{(1+f^2)}}\right) = Q\left(b - \frac{fr}{\sqrt{(1+f^2)}}\right).$$

$$\text{Whence } Pa = Qb - \frac{fr\,(P+Q)}{\sqrt{1+f^2}}.$$

And by supposing M to be on the other side of C, we find in like manner

$$Pa = Qb + \frac{fr\,(P+Q)}{\sqrt{1+f^2}}.$$

COR. 3. If f be small, we may neglect f^2: hence

$$Pa = Qb \pm fr(P + Q);$$

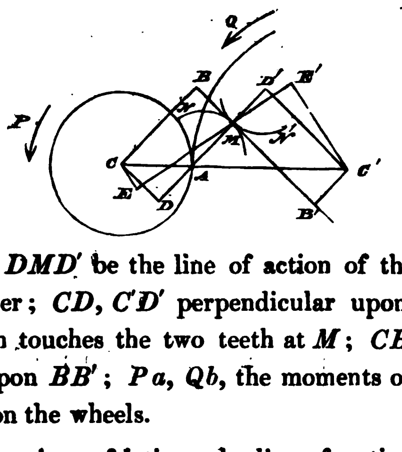
$$\text{whence } \frac{P}{Q} = \frac{b \pm fr}{a \mp fr}.$$

Ex. If $b = a$, $r = \frac{1}{5}a$, $f = \frac{1}{4}$, we have

$\frac{P}{Q} = \frac{20 \pm 1}{20 \mp 1}$; of which the values are $\frac{19}{21}$ and $\frac{21}{19}$, P and Q must not be more unequal than 19 and 21.

106. PROP. *When one wheel drives another by means of teeth; to find the conditions of equilibrium, friction at the surfaces of contact being taken into the account.*

Let a wheel with center C tend to turn a wheel with center C', a tooth NM of the first driving a tooth $N'M$ of



second. Let DMD' be the line of action of the one tooth upon the other; CD, $C'D'$ perpendicular upon DD'; BB' the line which touches the two teeth at M; CB, $C'B'$ perpendicular upon BB'; Pa, Qb, the moments of the forces which act upon the wheels.

When there is no friction, the line of action is perpendicular to BB', and if R be the action, we have

$$Pa = R \cdot CD, \quad Qb = R \cdot C'D';$$

$$\text{whence } Pa : Qb :: CD : C'D'.$$

But if friction act, the line of action is no longer necessarily normal to the surface, but may make with the normal

any angle not greater than the angle of sliding. If DME be the angle of sliding, EME' will be the line of action in the limiting case; and if CE, $C'E'$ be perpendiculars upon this line, we shall have, in that case,

$$Pa : Qb :: CE : C'E'.$$

Let $CA = a$, $C'A = b$, $CB = p$, $C'B' = q$, $CM = r$.

Let also the angle $MAC' = \alpha$, $CMD = \theta$, $DME = \phi$.

Then

$$CE = CM \sin CME = r \sin(\theta + \phi)$$
$$= r \sin\theta \cos\phi + r \cos\theta \sin\phi.$$

But $r \sin\theta = CD = a \sin\alpha$, $r \cos\theta = CB = p$,

$$\sin\phi = \frac{f}{\sqrt{1+f^2}}, \ \cos\phi = \frac{1}{\sqrt{1+f^2}}.$$

$$\text{Hence } CE = \frac{a \sin\alpha + fp}{\sqrt{1+f^2}}.$$

In the same manner we should find

$$C'E' = \frac{b \sin\alpha + fq}{\sqrt{1+f^2}}.$$

Hence $Pa : Qb :: a \sin\alpha + fp : b \sin\alpha + fq.$

$$\text{Whence } \frac{P}{Q} = \frac{1 + \dfrac{fp}{a \sin\alpha}}{1 + \dfrac{fq}{b \sin\alpha}},$$

which gives one limit of the ratio $\frac{P}{Q}$. And the other will be in like manner

$$\frac{P}{Q} = \frac{1 - \dfrac{fp}{a \sin \alpha}}{1 - \dfrac{fq}{b \sin \alpha}}.$$

If the tangent BB', which touches the teeth at the point of contact, pass on the same side of the centers C, C', instead of between them, we shall have in the same manner, for the first proportion,

$$Pa : Qb :: a \sin \alpha + fp : b \sin \alpha - fq.$$

COR. 1. For the one limit, when

$$\frac{P}{Q} = \frac{1 + \dfrac{fp}{a \sin \alpha}}{1 + \dfrac{fq}{b \sin \alpha}},$$

we have $P = Q \left\{ 1 + \dfrac{fp}{a \sin \alpha} - \dfrac{fq}{b \sin \alpha} - \dfrac{f^2 q^2}{b^2 \sin^2 \alpha} + \&c. \right\}$;

and neglecting f^2, as small,

$$P = Q + \frac{fQ}{\sin \alpha} \left(\frac{p}{a} - \frac{q}{b} \right),$$

of which the second term is the quantity by which friction requires P to be augmented, in order to produce a state bordering on motion.

COR. 2. If $AM = n$, we have

$$p = a \cos \alpha + n, \quad q = b \cos \alpha - n;$$

hence, on the suppositions of the last Corollary,

$$P = Q + \frac{fQ}{\sin \alpha} \left(\frac{n}{a} + \frac{n}{b} \right),$$

where Q is the force requisite for equilibrium when there is no friction, n the distance of the point of action from the common point of the wheels in the line of centers, α the angle which the line of action makes with the line of centers.

107. PROP. *When a cam drives a sliding piece; to find the conditions of equilibrium, friction at the surface of contact being taken into the account.*

This is a case of the last Proposition: we must suppose the center C' to be at an infinite distance, CA being perpendicular to the motion of the sliding piece. In this case, as before, n being AM, $p = a \cos \alpha + n$, $q = b \cos \alpha - n$.

$$Pa : Qb :: a \sin \alpha + f(a \cos \alpha + n) : b \sin \alpha + f(b \cos \alpha - n),$$

$$P : Q :: \sin \alpha + f \cos \alpha + \frac{fn}{a} : \sin \alpha + f \cos \alpha - \frac{fn}{b}.$$

And, since b is infinite,

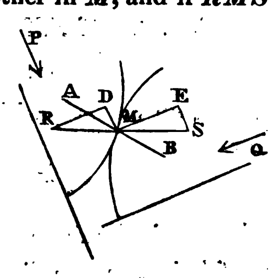
$$P : Q :: \sin \alpha + f \cos \alpha + \frac{fn}{a} : \sin \alpha + f \cos \alpha,$$

$$P = Q + \frac{Qfn}{a(\sin \alpha + f \cos \alpha)};$$

the second term is the quantity by which P must be increased in consequence of the friction.

108. PROP. *When one sliding piece drives another; to find the conditions of equilibrium, friction being taken into the account.*

If two sliding pieces touch each other in M, and if RMS be the line of action, P, Q the forces, RMD, SME (θ, η) the angles made by the line of action with the directions of sliding, we have $P = R \cos \theta$, $Q = R \cos \eta$; whence $P : Q :: \cos \theta : \cos \eta$, or $P \cos \eta = Q \cos \theta$.



Let AB be a normal to the surfaces at the point of contact, $AMD = \alpha$, $BMC = \beta$, the angles which this normal

makes with the directions of sliding, $\phi = AMR$, the angle of sliding for the friction. Then

$$P\cos(\beta - \phi) = Q\cos(\alpha + \phi):$$

$$P\cos\beta\cos\phi + P\sin\beta\sin\phi = Q\cos\alpha\cos\phi - Q\sin\alpha\sin\phi,$$

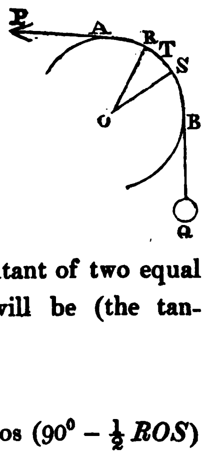
$$P\cos\beta + Pf\sin\beta = Q\cos\alpha - Qf\sin\alpha; \text{ whence}$$

$$\frac{P}{Q} = \frac{\cos\alpha - f\sin\alpha}{\cos\beta + f\sin\beta}:$$

and in like manner the other limit is found.

109. *When a cord passes along any curve surface, and is kept in equilibrium by forces at its two extremities: to find the conditions of equilibrium, friction being taken into the account.*

Let a force P balance a force Q, by means of a cord $PABQ$, which is in contact with the surface $ARSB$: let RS be a small arc, and RO, SO normals. The tensions of the cord at R and S act upon the small portion RS and press it against the curve. And if R, S represent the tensions at R and S, these will, at the limit, be equal,



and the resulting pressure will be the resultant of two equal forces: therefore (Mech. 16, Cor. 2,) it will be (the tangents at R and S meeting in T,)

$$2R\cos\tfrac{1}{2}RTS.$$

But $\cos\frac{1}{2}RTS = \cos\frac{1}{2}(180^0 - ROS) = \cos(90^0 - \frac{1}{2}ROS)$ $= \sin\frac{1}{2}ROS = \frac{1}{2}ROS$, because the angle is very small.

Let the small arc RS be called δs, and the normal RO, or SO be called ρ; then angle $ROS = \dfrac{\delta s}{\rho}$; and pressure of RS upon the curve $= 2\dfrac{R\delta s}{2\rho}$. Hence the friction of RS

is $\frac{fR\delta s}{\rho}$. Suppose the force S to be greater than the force R, and the equilibrium to be exactly preserved by the friction: therefore

$$S = R + \frac{fR\delta s}{\rho}; \quad \frac{S-R}{\delta s} = \frac{fR}{\rho}.$$

But ultimately, when the points R and S come together,

$$\frac{S-R}{\delta s} = \frac{dR}{ds}.$$

$$\text{Hence } \frac{dR}{ds} = \frac{fR}{\rho}: \quad \frac{1}{R}\frac{dR}{ds} = \frac{f}{\rho}.$$

If the curve be given, ρ is given, being the radius of curvature; and the expression may be integrated.

The integral is to be taken, so that at A, when $s = 0$, $R = P$.

COR. 1. If the curve be a circle, ρ is constant; hence, integrating,

$$\log \frac{R}{P} = \frac{fs}{\rho},$$

$$R = Pe^{\frac{fs}{\rho}}.$$

COR. 2. If the cord be wound round a cylinder and touch it in the half of a circumference, so that $\frac{s}{\rho} = \pi$, we have $R = Pe^{\pi}$.

In this case, if $f = 0.35$, we shall have very nearly, $e^{f\pi} = 3$.

Hence in the state bordering on motion, $R = 3P$; or $P = \frac{1}{3}Q$, for the least value of P; and in like manner, $P = 3Q$ for the greatest value of P.

COR. 3. If the cord make one turn and a half about the cylinder, these values will become

$$P = \frac{1}{27} Q, \quad P = 27 Q;$$

and if it make two turns and a half,

$$P = \frac{1}{243} Q, \quad P = 243 Q;$$

and thus at every additional turn the value of P must increase 9 fold, in order that the system may be in a state bordering on motion: and the force may be diminished in the same ratio, so as merely to sustain the weight.

Hence when a rope is wrapt round a cylinder, a very small force is sufficient to sustain a very great weight.

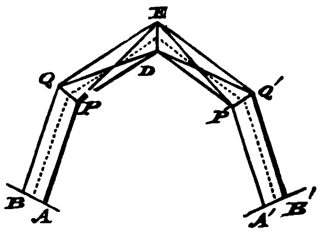
SECTION III. EQUILIBRIUM OF ARCHES IN PRACTICE.

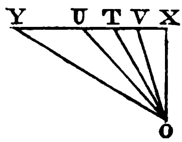
110. WE have in Chapter IV considered the conditions of equilibrium of an arch, supposing the surfaces of the arch-stones to offer no resistance to sliding. But in practice arches are constructed of stones or bricks which offer a great resistance to sliding; and therefore the conditions of their equilibrium are different from those formerly deduced. These conditions are given by the following Propositions.

111. PROP. *An isosceles polygonal frame, consisting of two pairs of beams touching each other in planes, is acted upon by given weights at the crown and at the haunches: to find the conditions of equilibrium, friction being taken into account.*

Let $AQDQ'A'$ be the frame, AQ, $A'Q'$ being one pair of beams, PE, $P'E$ another pair, and these beams touching

each other in the inclined planes PQ, $P'Q'$; and the vertical





plane DE, touching the abutments in the planes AB, $A'B'$. The beams themselves are supposed to be without weight, and to be acted upon by given weights at the crown DE, and at the haunches PQ, $P'Q'$: it is required to find the conditions of equilibrium.

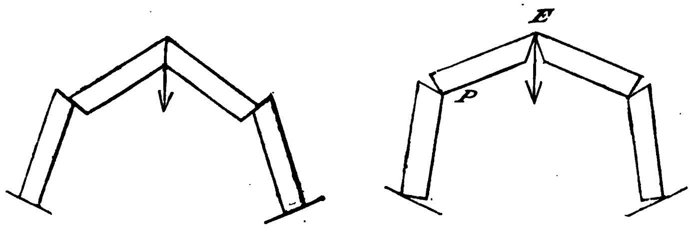
In the case in which the friction at the joints AB, PQ, is not brought into play, the pressure at these joints is perpendicular to the joints. And the pressures at the joints AB, PQ support the weight which acts at P or at Q. Hence the beam $APQB$ exerts its pressure in the direction perpendicular to AB, and therefore its length must be perpendicular to AB. For a like reason the length of the beam $PDEQ$ is perpendicular on EQ. And the same is true of the other half of the figure.

Draw OX, XY, vertical and horizontal, OY, OT, parallel to AP, PQ respectively. In the case in which friction does not act, the weight at P or Q is kept in equilibrium by those forces which are perpendicular to AB, to PQ, and to the horizontal line; that is, which are perpendicular to OY, to OT, and to TY: hence, the three forces are as the three sides of the triangle OYT, the weight at P or Q being as TY, and the pressure on the joint TQ being as OT. Also the weight at D or E may be conceived to be divided into two equal parts, and each half to act upon the other by a horizontal force at DE, which, along with the other forces,

keep it in equilibrium. Hence the half-weight at *D* or *E* is kept in equilibrium by three forces which are perpendicular to *DE*, *PQ*, and to the horizontal line; that is, which are perpendicular to *OX*, *OT*, and *XT*; hence the three forces are as the sides of the triangle *OXT*, the weight at *D* or *E* being as *XT*, and the pressure on the joint *PQ* being as *OT*. Hence the weight at *D* or *E* is to the weight at *P* or *Q* as *XT* to *TY*.

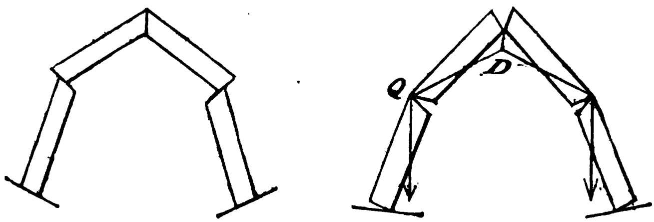
But let now either of these weights be increased. Then *XT* : *TY* no longer represents their proportion. Let, for instance, the weight at the *crown DE* be increased, and let *XU* : *UY* be the proportion of the weight at *DE* to that at *PQ*. Then joining *OU*, the pressure at the joint *PQ* must be perpendicular to *OU*, in order that there may be equilibrium. If *PE* be perpendicular to *OU*, and *AP* to *AB* or *OY*, the pressures of the beams must be exerted in the directions *AP*, *PE*.

Now the beam *PQDE* can exert a force in the direction *PE*, friction being supposed to act, if (*PD* being perpendicular to *PQ*) the angle *DPE* is less than the angle of sliding, for the friction. If this be not the case, sliding will take place at the joint *PQ*, the beam *PQDE* sliding inwards, and the beam *ABPQ* sliding outwards, as in the figure.



It is also requisite that the line *EP* do not fall without the actual surface of the joint *PQ*; for if it do, the beam *PE* will turn round the inner angle of the joint *PQ*, as in the second of the above figures.

In like manner let the weight at the *haunches* PQ, $P'Q'$ be increased, and let $XV : VY$ be the proportion of the weight at DE to the weight at PQ; and join OV. Then the pressure at the joint PQ must be perpendicular to OV, in order that there may be equilibrium. If QD (page 100) be perpendicular to OV, and QB to AB, the pressures of the beams must be exerted in the directions BQ, QD. And (QE being perpendicular to QP) if the angle DQE be less than the angle of sliding for the friction, there will be equilibrium. But if not, the beam $PQDE$ will slide outwards.



It is also requisite that the line of pressure QD do not fall without the actual surface of the joint ED. If it do, the beam PQ will turn round the inner angle of the joint DC, and the outer angle of the joint PQ, as in the second of the above figures.

Thus the conditions of equilibrium of the frame in question are these: the polygon $APEP'A'$ or $BQDQ'B'$ (p. 100) being constructed, the form of which depends upon the weights at the crown and at the haunches, it is requisite, in order to prevent sliding, that the sides of this polygon meet the joints in such directions that the angles with the normal on either side do not exceed the angle of sliding.

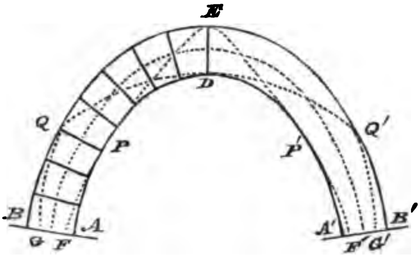
Moreover it is requisite for equilibrium that the line $APEP'A'$ or $BQDQ'B'$, in which the pressures are exerted fall within the limits of the planes in which the beams touch each other. If this line fall below the points P, P', the beams will turn round the points P, P', the haunches

rising and the crown descending: and if the line of pressures fall below the point D, the beams will turn round point D, the haunches descending and the crown rising.

Def. *The line of pressure* is the polygon formed by drawing the lines in which the pressures at the joints of an arch or frame act.

112. Prop. *In an arch acted upon by weights at all the joints, friction being taken into account, to show the consequences of increasing the load upon the crown and upon the haunches.*

Let AEA' be an isosceles arch in equilibrium. In any



case we may draw the line of pressure, without friction, by beginning at the crown, and forming a polygon which shall have its sides perpendicular to the joints, at the points at which the weights act.

It has appeared in the last Proposition, that in an arch consisting of four voussoirs, if we add a weight at the crown, the line of pressure ascends at the crown and descends at the haunches; and if we add weight at the haunches, the line of pressure ascends at the haunches and descends at the crown. And by similar reasoning, the same is the case in an arch consisting of a greater number of voussoirs. Hence by adding weight at the crown, the line of pressure assumes the form $FPEP'F'$; and by increasing the weight at the crown, this line either ascends above the arch-ring at E, or

descends below it at P, P'; and in either case the arch breaks, the crown descending, and the haunches ascending. Again, by adding weight at the haunches, the line of pressure assumes the form $GQDQ'G'$; and by increasing the weight at the haunches, this line either descends below the arch-ring at D, or ascends above it at Q, Q'; and in either case the arch breaks, the haunches descending and the crown ascending.

It is supposed in this reasoning that the line of pressure, in the course of the above changes, every where makes with the normals to the joints an angle less than the sliding angle. If it do not, sliding will take place at the joints where the angle transgresses this limit.

Cor. Before the line of pressure falls beyond the intrados at P, it comes very near to P. In this case the pressure on the parts of the joints near to P will be very great; and it may be so great that the material may yield and be crushed. Thus by loading the crown, there is a tendency to crush the arch-stones in the inside under the haunches.

In like manner by loading the haunches, there is a tendency to crush the arch-stones in the inside at the crown.

113. The two conditions above stated are the conditions of equilibrium for all machines and structures whatever, constructed of masses merely in contact; (that is, not connected by pins, mortises, joggles, cement, and the like;) namely,

1°. *That the lines of pressure pass within the limits of the surfaces which are in contact.*

2°. *That the lines of pressure meet the surfaces of contact in such directions that the angle which they make with the normal to each surface is less than the sliding angle.*

These conditions are necessary and sufficient, supposing the material not to yield.

The pressures at the surface of each mass, (simple or compound,) will act in such directions that their resultant is in the vertical line, passing through the center of gravity of the mass; and their magnitude will be determined by the condition that their resultant is equal and opposite to the weight of the mass. Whether the material will yield under the action of their forces, will depend upon the strength of the materials; a problem which we do not consider at present.

SECTION IV. THE EQUILIBRIUM OF OBLIQUE ARCHES IN PRACTICE.

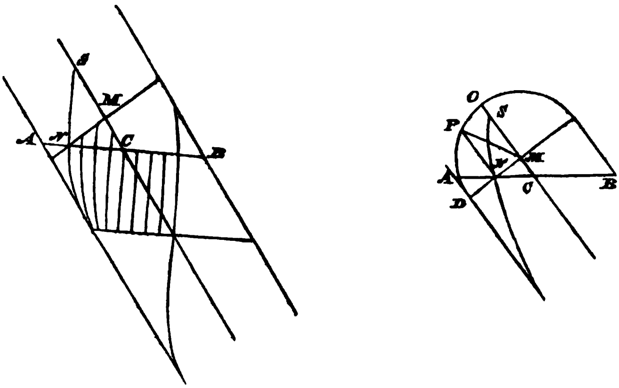
114. We formerly deduced the form of the courses of an oblique arch, which is requisite in order that there may be equilibrium, the parts being perfectly smooth. But the form of the courses there deduced is such as could not be conveniently employed in practice, since the courses are wider at one end than the other; and therefore could not be built of a number of equal parallelopipeds, such as are bricks and squared stones. But we may, by means of bricks or squared stones, build an arch in which the position of the courses approximates to that of the equilibrated courses; and an arch so built will, within certain limits, be kept in equilibrium by friction.

The soffit of an oblique arch is a surface described by a line which is always parallel with itself. If we draw upon this surface a line making at any point equal angles with the describing line, this line is a *spiral*. If two such lines be drawn near each other, their distance along the describing line will be everywhere the same. And therefore the length of a perpendicular from one upon the other will be always the same.

Hence if the soffit be divided by such lines into narrow courses of equal breadth, these courses may be built of equal bricks or squared stones, except so far as the plane surface of one brick or stone deviates from the corresponding curved surface of the soffit, which deviation is small. In practice oblique arches are thus built; and the statical conditions of such arches are given by the following Propositions.

115. PROB. *To find the horizontal projections of spiral courses in an oblique arch, the soffit being a cylinder of which the transverse section is a semicircle.*

Let CM be the axis, MN the transverse section, OP



the arc of the transverse section measured from the highest point; S the point where the spiral line crosses the summit line of the cylinder. If the cylindrical surface be unwrapt into a plane, the spiral will become a straight line making a given angle with SC. Hence if γ be this angle,

$$OP = SM \tan\gamma.$$

Let $SM = x$, $MN = y$, $DM = b$; then $OP = x\tan\gamma$,

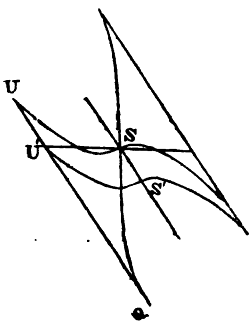
$$y = DM \sin\frac{OP}{DM} = b\sin\frac{x\tan\gamma}{b}; \quad \frac{y}{b} = \sin\frac{x\tan\gamma}{b}.$$

Hence the horizontal projection of the spiral is the *figure of sines;* namely, the figure in which the abscissa being as the arc, the ordinate is as its sine.

And the projections of the courses will be determined by a series of such figures drawn through successive points of the summit line.

116. PROP. *An oblique arch being built with spiral courses at any angle: to find the lines of pressure.*

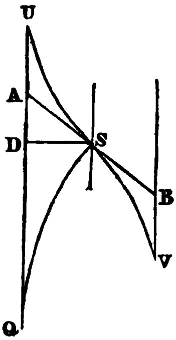
The spiral on the cylindrical soffit makes everywhere an angle γ with lines parallel to the axis. But the lines of pressure (friction being rejected) will be everywhere perpendicular to the bed-joints. Hence the lines of pressure will everywhere make an angle $90^{\circ} - \gamma$ with the lines parallel to the axis; and therefore will also form spiral lines on the soffit. Let SQ be one of the bed-joint spirals; and SU a spiral perpendicular to SQ at S; then SU will be the line of pressure; and in like manner other spirals, as $S'U'$, will be the lines of pressure for other sections of the arc.



117. One common mode of practically constructing oblique arches is to make the bed-joints perpendicular to the roadway, in the summit line of the soffit. If this be the case, the line of pressure SU (without friction) touches the face SA (see next page) at the summit S; and one half of it falls outside the structure. Hence the equilibrium depends, in this case, upon friction and cohesion; which must act so as to throw the actual line of pressure within the structure.

PROP. *To find how far the line of pressure, without friction, in this case falls beyond the structure.*

Let β be the obliquity of the arch, that is, the angle which the axis of the arch makes with a perpendicular to the roadway. Hence in this case $\gamma = \beta$. And by Art. 115,



$$\text{when } y = b, \quad \frac{x \tan \gamma}{b} = \frac{\pi}{2};$$

hence, $x = DQ = \dfrac{\pi b}{2} \cot \gamma = \dfrac{\pi b}{2} \cot \beta$,

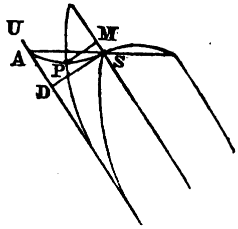
and in like manner

$$DU = \frac{\pi b}{2} \tan \beta;$$

also $DA = b \tan \beta$; hence $AU = b \tan \beta \left(\dfrac{\pi}{2} - 1\right)$.

118. But the oblique arch may be constructed, with other angles for the courses than that thus determined. If γ be made less than β, the line of pressure, without friction, SU, will not fall so much without the structure on the side SA; but then the line of pressure SV will then fall without the structure on the side SB. The actual line of pressure must coincide nearly with the line of the roadway AB, in order that the pressure may be thrown upon the abutments.

This will be most nearly the case if U and V coincide with A and B.



In this case, $DU = \dfrac{\pi b}{2} \tan \gamma$,

$$DA = b \tan \beta.$$

$$\text{Hence } \frac{\pi b}{2} \tan \gamma = b \tan \beta,$$

$$\tan \gamma = \frac{2}{\pi} \tan \beta.$$

which determines γ, when β is known. Hence it appears that if β be small, γ is about $\frac{2}{3}$ of β.

119. Prop. *In this case, to find where the joints are perpendicular to the face.*

The joints are perpendicular to the face when the line of pressure (without friction) is parallel to the face. The equation to the curve SP is (Art. 115, 116.)

$$y = b \sin \frac{x \cot \gamma}{b}.$$

Hence $\frac{dy}{dx} = \cot \gamma \cdot \cos \frac{x \cot \gamma}{b}$.

But the curve is parallel to the face AS when

$$\frac{dy}{dx} = \cot \beta.$$

Hence $\cot \beta = \cot \gamma \cdot \cos \frac{x \cot \gamma}{b}$;

or $\frac{\tan \gamma}{\tan \beta} = \cos \frac{x}{b \tan \gamma}$:

and putting for $\tan \gamma$, $\frac{2}{\pi} \tan \beta$, found above,

$$\frac{2}{\pi} = \cos \frac{\pi}{2} \frac{x}{b \tan \beta}.$$

Let $\frac{2}{\pi} = \cos a$: then $a = \frac{\pi}{2} \frac{x}{b \tan \beta}$ at the point sought,

$$\frac{x}{b \tan \beta} = \frac{2a}{\pi}; \quad \frac{x}{b \tan \gamma} = a; \quad y = b \sin a.$$

It appears by the tables that $a = 50\frac{1}{2}^0$ nearly: and $\sin a = .77$.

If the transverse section of the soffit were a semicircle, the soffit, at the point sought, would make with the vertical

plane an angle of which the tangent would be $\frac{y}{b}$, or sin α; that is, this angle would be $37\frac{1}{2}^{\circ}$.

Hence it appears that the point where the courses are perpendicular to the face of the arch is at a distance from the spring, of about one-fifth of the whole curve of the arch.

120. If friction be supposed to act, and if it be so great that it can act in the line SA parallel to the face of the arch, the plane represents the actual line of pressure, and the equilibrium will subsist with spiral courses. In this case the angle which the line of pressure makes with the courses (the courses being perpendicular to the face at the summit line,) is, at the spring, the complement of β the obliquity. Hence such an arch cannot stand by friction, if β be greater than the angle of sliding by friction. If this is so, the archstones near the acute spring on each face will slide outwards.

If the courses be spiral, and make an angle γ with the summit line, they will, at the spring, make an angle γ with the horizontal line; and therefore the line of pressure (with friction) at the spring, will make an angle γ with the perpendicular to the bed-joint. Hence there will be equilibrium, with friction, if γ be less than the sliding angle.

By making γ small, we may prevent sliding. But by doing so, we tend to throw the line of pressure without the structure; and then the structure breaks by turning round the *back-joints*. If the ring of arch-stones on the face be so long in the direction of the axis, as that the line of pressure, without friction, falls within the space occupied by the stones, the arch will stand without friction.

121. Prop. *Given the number of courses: to find the breadth of any course on the face.*

Let there be n courses between A and C (fig. p. 106.)

By Art. 115,

$$SC = SM + MC = x + y \tan\beta$$

$$= \frac{b}{\tan\gamma} \text{ arc } \left(\sin = \frac{y}{b}\right) + y \tan\beta.$$

And when N comes to A, this is

$$= \frac{\pi b}{2 \tan\gamma} + b \tan\beta.$$

If points S be taken at equal intervals along CS, we shall have the projections of the spirals which divide the soffit into courses of equal breadth. Let c be one of these intervals. Then

$$nc = \frac{\pi b}{2 \tan\gamma} + b \tan\beta; \text{ whence } c.$$

And if for any two successive distances mc and $(m+1)c$, the projected spirals be drawn, $(SN, S'N')$ their intersections (N, N') with AC will determine the projection of the interval of the intrados corresponding to this course.

CHAPTER VI.

STABLE AND UNSTABLE EQUILIBRIUM.

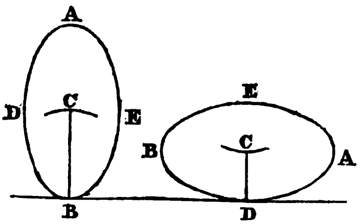
122. Def. *The equilibrium of a system is* stable, *when the system, being slightly deranged from the position of equilibrium, has a tendency to return to it. The equilibrium is* unstable, *when the system, being slightly deranged from the position of equilibrium, has a tendency to recede from it further.*

Thus in Mech. 46, it appeared that if a rigid body be moveable about a point, it has a position of equilibrium when the center of gravity is at the lowest, and also when it is at the highest point of the circle which it describes in a vertical plane. It appeared also that if the body be slightly deranged from the former position, it has a tendency to return to it; but that if the body be slightly deranged from the latter position, it has a tendency to recede further from it. Hence in this case, there is a position of stable equilibrium when the center of gravity is at the lowest point, and a position of unstable equilibrium when the center of gravity is at the highest point.

123. It has been shewn already (Art. 47), that in the position of equilibrium, the center of gravity is always at one of the lowest or highest of its possible positions. It is true in general (as we have just seen it to be in a particular case) that the position in which the center of gravity is at one of its lowest positions the equilibrium is stable, and when it is at one of the highest positions the equilibrium is unstable.

But we cannot prove this in general without determining the motion which would take place in virtue of a small derangement; which is a problem belonging to Dynamics.

As an example, suppose an egg-shaped body to stand



on a horizontal plane: it will be in equilibrium when its axis passing through the center of gravity, and also through the point of contact with the plane, is vertical. And if the figure be symmetrical, this will take place when the shortest axis is vertical, and when the longest axis is vertical. But if we suppose the figure to change its position by rolling through a small space in the neighbourhood of either of these positions, the center of gravity will, in the former case, describe a curve convex downwards; and in the latter case, a curve convex upwards. The center of gravity, is in the former case, in one of its lowest, and in the latter, in one of its highest positions: and the former is a position of stable, the latter a position of unstable equilibrium.

124. Prop. *If by continuous changes a system pass into several positions of equilibrium, these are alternately stable and unstable.*

Let a rigid system, by continuous change of situations, pass from one position of stable equilibrium, A, to another position of stable equilibrium, C. Then when the system is slightly deranged from the condition A towards the condition C, its tendency is to return to the position A, because A is a position of stable equilibrium; and when it is slightly de-

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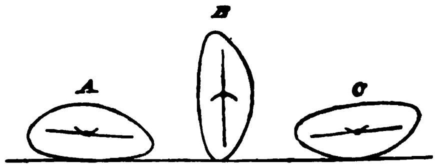
ranged from the position C towards the position A, that is,

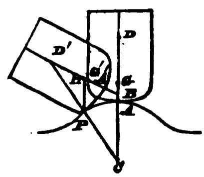


before in its continuous change it reaches the position C, its tendency is to fall to the position C, because C is a position of stable equilibrium. Thus in passing from the position A to the position C, the tendency is first to fall to the position A, and afterwards to fall to the position C. And the amount of this tendency changes continuously, by the continuous change of position. Hence this tendency will pass from the direction CA to the direction AC by passing through a point in which it vanishes. That is, there will be an intermediate position B, in which there is no tendency to fall either to the position A or to the position C; that is, there is a position of equilibrium B. And from A to B the tendency is to fall to the position A, and from B to C the tendency is to fall to the position C; therefore, in both cases the tendency is to fall from the position B, when we suppose a position slightly deviating from B: that is, B is a position of unstable equilibrium. Therefore, between the two positions of stable equilibrium A, C, there is a position of unstable equilibrium. And the same may be proved in any other case. Therefore, &c. Q. E. D.

125. PROP. *A heavy body of any form rests upon a surface of any form: to find in what case the equilibrium will be stable.*

The surfaces are supposed perfectly smooth, and hence at the point of contact they will be horizontal. Also whatever be the form of the surfaces, the section, or small por-

tion of each in the neighbourhood of the point of contact, made by the vertical plane in which the derangement takes place, may be conceived to be a circle; the radius of which will be the radius of curvature of the corresponding portion of the surface.



Let A be the point of contact in the position of equilibrium, C the center of curvature of the section of the supporting surface, D the center of curvature of the section of the lower part of the body; G the center of gravity of the body. Let the body roll, so that the point of contact becomes P, A coming to A', G to G', D to D': then CP will pass through D'. Draw PH vertical, meeting $A'D'$ in H.

The arc $A'P = AP$, being the same arc.

Hence

$$\text{angle } A'D'P : \text{angle } ACP :: \frac{A'P}{A'D'} : \frac{AP}{AC} :: AC : AD.$$

But $PH : HD' :: \sin PD'H : \sin HPD'$,

$:: \sin A'D'P : \sin ACP$, by parallels,

$:: \text{angle } A'D'P : \text{angle } ACP$, because the angles are very small,

$:: AC : AD.$

$$\text{Hence } PH + HD' : HD' :: AC + AD : AD;$$

$$\text{whence } HD' = (PH + HD')\,\frac{AD}{AC + AD}.$$

Now if the point G' fall below H the equilibrium is stable; for in that case the whole weight of the body, which may be collected at G', tends to turn the body back to its position of equilibrium. That is, the equilibrium is stable if $HD' < G'D'$.

But since the derangement must be small, the ultimate values must be taken; in which case $PH + HD' = AD$; hence in this case,

$$HD' = \frac{AD^2}{AC + AD};$$

and, since $G'D' = GD$, the equilibrium is stable

$$\text{if } GD > \frac{AD^2}{AC + HD}.$$

If GD be less than this, the equilibrium is unstable; if it be equal to this, the equilibrium is *indifferent*, that is, this condition does not determine it either to be stable or unstable. For a *small* derangement there is no tendency either to return to the original position, or to recede further from it.

Cor. Since $GD = AD - AG$, the equilibrium is stable,

$$\text{if } AD - AG > \frac{AD^2}{AC + AD}; \text{ or if } AG < \frac{AC \cdot AD}{AC + AD}.$$

Ex. A body, the lower part of which has a radius of 1 foot, rests on a body, the upper part of which has a radius of 2 feet: to determine how high the center of gravity may be, so as to preserve the stability of the equilibrium.

$$\text{In this case } AG < \frac{AC \cdot AD}{AC + AD} = \frac{2 \times 1}{2 + 1} = \frac{2}{3}.$$

If the height of centre of gravity be less than $\frac{2}{3}$ of a foot above the base, the equilibrium is stable; if the height be greater, the equilibrium is unstable; if the height be $\frac{2}{3}$ of a foot, the equilibrium is indifferent.

126. The stability of a system may be *measured* by the moment which acts to bring back the system to the position of equilibrium, when it is deranged through a very small angle.

Prop. *In the last Proposition, to find the measure of the stability.*

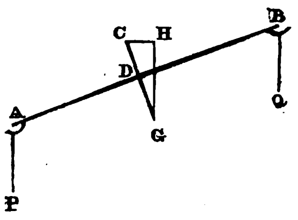
If W be the weight of the body, θ the angle which $D'A'$ (last Art.) makes with the vertical. The moment which tends to turn the body back to the position DA is $W.H'G'.\sin\theta$. Hence the stability is as $W.HG'$; that is, as $W(D'G' - D'H)$; or, putting DG for $D'G'$, and for $D'H$ its value, the stability is as

$$W\left(DG - \frac{AD^2}{AC+AD}\right), \text{ or as } W\left(AD - AG - \frac{AD^2}{AC+AD}\right),$$

$$\text{or as } W\left(\frac{AC.AD}{AC+AD} - AG\right).$$

127. Prop. *To find the stability of a balance.*

In a balance, let the arms $DA = DB = a$; the perpendicular from the point of suspension upon AB, $= CD = c$; the distance from the point of suspension to the centre of gravity $CG = b$; GH vertical, and the angle $CGH = \theta$; the weight of the beam $= B$; the weight at A or $B = Q$. Then the equal weights P and Q may be conceived to be collected at D, and the weight B at G. Therefore the moment round C is



$$2Qc\sin\theta + Bb\sin\theta = (2Qc + Bb)\sin\theta.$$

Hence the moment for a given angle is as $2Qc + Bb$; and Q being given, the stability is greater as c is greater, and as Bb is greater.

Cor. 1. The *sensibility* of a balance may be measured by the smallness of the weight, which, when it is in equilibrium, will make it deviate through a given angle. Let

the weights at A and B be Q and $Q+x$; then, by the equilibrium of the lever,

$$(Q+x)(a\cos\theta - c\sin\theta) = Q(a\cos\theta + c\sin\theta) + Bb\sin\theta;$$

$$\text{or } x(a\cos\theta - c\sin\theta) = 2Qc\sin\theta + Bb\sin\theta;$$

$$\text{whence } x = \frac{2Qc + Bb}{a\cos\theta - c\sin\theta}.\sin\theta.$$

If θ be very small, we may put θ for $\sin\theta$, 1 for $\cos\theta$, and neglect $c\sin\theta$. Hence we have

$$x = \frac{2Qc + Bb}{a}\theta.$$

And therefore the sensibility, which is inversely as x for a given angle θ, is as $\dfrac{a}{2Qc + Bb}$.

Cor. 2. Hence by increasing the arm a, we increase the sensibility without diminishing the stability.

128. The equilibrium is also stable when the position of equilibrium is not disturbed, or only slightly disturbed, by increasing in some degree one or other of the forces which act to destroy the equilibrium in opposite directions.

Prop. *In cases of equilibrium in which friction acts, the equilibrium is always stable.*

For the weights and forces having the relations which equilibrium without friction requires, if friction be introduced, certain additional forces are required to destroy the equilibrium, either in one direction or in the other. Therefore, within these limits, the equilibrium will not be disturbed by additional forces, and is therefore stable.

CHAPTER VII.

THE ELASTICITY AND FLEXURE OF SOLID MATERIALS.

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129. All solid substances, as wood, stone, metals, &c. are susceptible of some compression and extension. This compression and extension are greater as the forces producing them are greater; and when the forces produce a compression or extension greater than the texture of the substance can bear, the bodies are crushed or broken. We shall here find the change of figure of such bodies when they are compressed under given circumstances.

We shall suppose that all solid bodies may be considered as made up of elastic fibres, capable of extension and compression. We shall also suppose, that the resistance to extension is proportional to the extension in each given fibre, and the same of compression. We shall further assume, that the resistance to extension and to compression are the same in the same fibre.

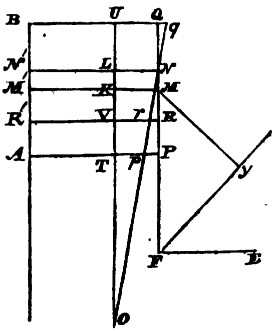
These principles would follow if we were to suppose the particles of bodies to be kept in equilibrium by their mutual forces in the natural state of the body; and the change to be small, which they undergo by the action of any force. In this case it might be proved that the displacement of a given particle would be ultimately as the force which produces it.

130. Let a prismatic body be extended or compressed in the direction of its length by a force t; and let the in-

crease or diminution of the length for a given length be proportional to the extending or compressing force. Then for twice, thrice, &c. the length, the increase or diminution of length will be twice, thrice, &c. as great; since each equal length will be equally compressed by the same force. Hence if l be the length, and x the increase of length, we shall have x proportional to lt; and if E be a constant quantity, suitably taken, $x = \frac{lt}{E}$. Also if $t = E$, $x = l$; that is, if the extending force be E the extension is equal to the original length. The quantity E is the force which is requisite to extend l to double its length, the law of extension continuing the same.

The force t may be expressed by the length of a column of the prismatic body, namely, by the weight of such a column; and on this supposition, E is also a certain length. The quantity E is greater, in proportion as a greater force is requisite to produce an equal extension or compression in the body; and E is called the *modulus of elasticity* for the substance in question.

131. When a solid body is acted on by any force, it may be partly extended and partly compressed. Thus, let a mass $ABQP$ be acted upon by a force F, compressing it in the direction EF. The surface PNQ may be brought into the direction pNq; in this case, all the fibres RR' which are on one side of N are shortened; all those on the other side of N are lengthened. NN' remains the same as in the natural state. N is called the



neutral point, and the line which separates the parts of

a transverse section of the body which are compressed from those which are elongated, is called the *neutral line* of that section.

132. PROP. *When a rectangular prismatic mass is compressed by a force parallel to the direction of the axis at a given distance: to find the neutral line.*

Let AB, fig. in last Article, be the rectangular base of the mass, MM' its axis. And let the slice $UTPQ$ be compressed so as to assume the form $UTpq$, N being the neutral line. Then any fibre parallel to the axis, as VR, is compressed so that its length becomes Vr: and by the supposition, if t be the force compressing it, E the modulus of elasticity, as in Article 130; we shall have

$$Rr = VR\,\frac{t}{E}; \text{ and hence } t = E\,\frac{Rr}{VR}.$$

Let $PM = MQ = a$, $MF = h$, $MR = x$, and the breadth of the beam perpendicular to the plane $MFE = b$; $MN = n$, whence $RN = n + x$; force at $F = f$.

Also let UT and qp meet in O, and let $OK = \rho$.

Hence

$$\frac{Rr}{VR} = \frac{Rr}{NL} = \frac{NR}{OL} = \frac{n+x}{n+\rho}.$$

And the force of VR, supposing its breadth and thickness each 1, is

$$t = E\,.\frac{Rr}{VR} = E\,.\frac{n+x}{n+\rho}.$$

Hence if we take a very thin portion, of which the thickness (in direction NR) is δx and breadth b, its force is

$$E\,.\frac{n+x}{n+\rho}\,.\,b\delta x,$$

and this is the increment of the force exerted at R corresponding to δx. When x is negative and greater than n,

this is negative; and accordingly, the compression for that part becomes extension.

The forces which keep each other in equilibrium are the force f acting in EF, and the elementary forces of all the fibres VR. And hence, by Mech. 33, we must have, 1st, the force f equal to all the forces

$$E \cdot \frac{n+x}{n+\rho} b\delta x;$$

and 2nd, the moment of the force f about N equal to the moments of all the forces

$$E \cdot \frac{n+x}{n+\rho} b\delta x \text{ about } N.$$

Also the aggregate of all the forces will be found by taking the coefficients of δx, in the expressions so found, and the integrals of these differential coefficients from

$$x = -a, \text{ to } x = a.$$

Hence we have

$$f = \int_x E \frac{n+x}{\rho+n},$$

$$f(h+n) = \int_x E \frac{(n+x)^2}{\rho+n}.$$

Integrating between the proper limits, ρ being constant,

$$f = E \cdot \frac{2nab}{\rho+n},$$

$$f(h+n) = E \cdot \frac{2n^2ab + \frac{2}{3}a^3b}{\rho+n}.$$

Dividing, we have

$$h+n = n + \frac{a^2}{3n}; \quad h = \frac{a^2}{3n};$$

$$\therefore n = \frac{a^2}{3h} = \frac{(2a)^2}{12h}; \text{ or } MN = \frac{PQ^2}{12MF}.$$

COR. 1. If $MF = \frac{1}{3}MP$, or $h = \frac{1}{3}a$, $n = a$, the neutral point is in the surface, and the whole beam is compressed.

If $MF > \frac{1}{3}MP$, the neutral point is beyond the surface.

COR. 2. From the above equations we have

$$\rho + n = \frac{E}{f} . 2nab = \frac{E}{f} . \frac{2a^3b}{3h} .$$

And $\rho + n$ is the radius of curvature of the neutral line NN' at N. Let the force f be equivalent to a length F of the prism; then $f = 2Fab$; and we have

$$\rho + n = \frac{E}{F}\frac{a^2}{3h}; \text{ or } NO = \frac{E}{F} . \frac{PQ^2}{12MF} .$$

133. PROP. *When a rectangular prism is acted upon by any force in any direction: to find the neutral point at any point.*

Let a force f, fig. in Art. 131, act in the line yF on a prism $ABPQ$. The force will produce the same effect as if it acted at F, a point in QP. Let the angle MFy at $F = a$. The force may be resolved into $f \cos a$ in QP, and $f \sin a$ perpendicular to QP. Of these, the former is resisted by the lateral resistance of the materials, and produces no compression. The latter produces a compression as in last Article. Hence retaining the denominations of last Article, calling MF, h, and putting $f \sin a$ for f, we have

$$f \sin a = E . \frac{2nab}{\rho + n};$$

$$f(h + n) \sin a = E . \frac{2n^2ab + \frac{2}{3}a^3b}{\rho + n} .$$

And hence $h + n = n + \dfrac{a^2}{3h}$; $\therefore$ $n = \dfrac{a^2}{3h}$.

COR. 1. We have also

$$\rho + n = \frac{E}{f \sin \alpha} \cdot 2nab = \frac{E}{f \sin \alpha} \cdot \frac{2a^3 b}{3h}$$

$$= \frac{E}{f} \cdot \frac{2a^3 b}{3h \sin \alpha} = \frac{2Ea^3 b}{3fk},$$

if $k = My = h \sin \alpha$, the perpendicular on the direction of the force from the axis.

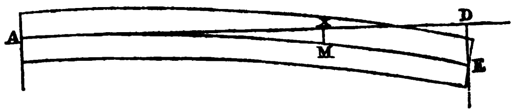
COR. 2. If as before $f = 2Fab$,

$$\text{rad. of curv.} = \rho + n = \frac{Ea^2}{3Fk}.$$

COR. 3. If the force act perpendicularly to the axis, h is infinite, $n = 0$, and the neutral point is in the axis.

134. PROP. *When a rectangular prismatic beam fixt at one end, is made to deviate a little from a straight line by the action of a given force perpendicular to it: to find the deflexion.*

Since the force is perpendicular to the beam, and the beam is nearly a straight line, we may, by Cor. 3 of last Article, suppose the neutral point to be everywhere coincident with the axis. Let AME, Art. 131, represent the axis, bent by a force acting perpendicularly to AD, its original position.



And let XM be the ordinate at any point, also perpendicular to AD. $AX = x$, $XM = y$. And since the curve is nearly a straight line, $\frac{dy}{dx}$ is small: hence the radius of curvature

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}} \text{ is } = \frac{1}{\frac{d^2 y}{dx^2}}, \text{ nearly.}$$

But by Cor. 2. to last Art. if $AD = l$, $k = DX = l - x$,

$$\text{rad. of curv.} = \frac{E}{F} \cdot \frac{a^2}{3(l-x)};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{F}{E} \cdot \frac{3(l-x)}{a^2}.$$

Integrate with respect to x, observing that $\frac{dy}{dx} = 0$, when $x = 0$:

$$\therefore \frac{dy}{dx} = \frac{F}{E} \cdot \frac{3lx - \frac{3}{2}x^2}{a^2}.$$

Integrate again: observing that $y = 0$, when $x = 0$.

$$y = \frac{F}{E} \cdot \frac{\frac{3}{2}lx^2 - \frac{1}{2}x^3}{a^3}.$$

And if the whole deflexion $DE = \delta$, making $x = l$,

$$\delta = \frac{F}{E} \cdot \frac{l^3}{a^3}.$$

Cor. 1. If we put for F its value $\frac{f}{2ab}$, (Art. 133,) we have

$$\delta = \frac{fl^3}{2Ea^3b}.$$

Hence it appears that for a given breadth and thickness the deflexion is as the force and cube of the length.

And for a given force and length the deflexion is inversely as the breadth and cube of the thickness.

Cor. 2. Let the direction of the tangent at E make an angle θ with the tangent at A. Then θ may be called the *angular deflexion.*

And $\frac{dy}{dx} = \tan\theta$; hence, putting l for x in the value of $\frac{dy}{dx}$, at the extremity, $\tan\theta = \frac{F}{E} \cdot \frac{3l^2}{2a^2} = \frac{3fl^2}{4a^3b}$.

The extreme angular deflexion is as the force and square of the length.

135. Prop. *When a rectangular prismatic beam, fixt in a horizontal position, is bent by its own weight (its thickness being vertical): to find the deflexion.*

In Art. 133, Cor. 2; put, for Fk the moment of the force which bends the beam, $(l-x)\dfrac{l-x}{2}=\frac{1}{2}(l-x)^2$; and for the rad. of curv. $\dfrac{1}{\frac{d^2y}{dx^2}}$.

Hence we have

$$\frac{d^2y}{dx^2}=\frac{3(l-x)^2}{2Ea^3};\quad \frac{dy}{dx}=\frac{l^3-(l-x)^3}{2Ea^3};$$

$$y=\frac{l^3x+\frac{1}{4}(l-x)^4-\frac{1}{4}l^4}{2Ea^3};$$

and the whole deflexion $\delta=\dfrac{3l^4}{8Ea^4}$.

Cor. In this and the last Article, δ being observed, E may be found.

136. Prop. *When an isosceles triangular prism is acted upon by any force in any direction: to find the neutral point at any part.*

The force is supposed to act in the plane which bisects the vertical angle of the isosceles triangle. Let $ABQP$, Art. 130, be this plane, the vertex of the triangle being at P, and its base at Q.

Let $OT=\rho$, $TV=x$, $TL=n$, $TU=a$, $PF=h$, $MFy=\alpha$, the force $=f$, modulus of elasticity $=E$.

As before, in Art. 132, we shall have the force of a single fibre at $R=E\dfrac{NR}{OL}=E\dfrac{n-x}{\rho+n}$.

peripheral

And whatever be the form of the section perpendicular to the plane $ABQP$, if y be the ordinate of this section perpendicular to the line PQ, we shall have for the elementary force exerted at R,

$$E\frac{n-x}{\rho+n}y\delta x.$$

And by the same reasoning as in Art. 134,

$$f\sin a = E\int_x \frac{n-x}{\rho+n}y,$$

$$f(h+n)\sin a = E\int_x \frac{(n-x)^2}{\rho+n}y.$$

In the case of the triangle, $y = mx$, m being a constant quantity. And integrating from $x = 0$ to $x = a$,

$$f\sin a = \frac{Em}{\rho+n}\cdot\left(\frac{1}{2}na^2 - \frac{1}{3}a^3\right),$$

$$f(h+n)\sin a = \frac{Em}{\rho+n}\cdot\left(\frac{1}{2}n^2a^2 - \frac{2}{3}na^3 + \frac{a^4}{4}\right);$$

$$\therefore\ h+n = \frac{6n^2 - 8na + 3a^2}{6n-4a}$$

$$= \frac{3a^2 - 4an}{6n-4a} + n;$$

$$\therefore\ h = \frac{3a^2 - 4an}{6n-4a};$$

$$\therefore\ n = \frac{3a^2 + 4ah}{4a+6h}.$$

Cor. 1. If $h = 0$, or the force act at P, $n = \frac{3}{4}a$.

Cor. 2. If the force act perpendicularly to the prism, h is infinite, and $n = \frac{2a}{3}$.

COR. 3. If the force act above P, h will be negative. Thus if the force act at Q, $h = -a$, $n = \frac{a}{2}$.

COR. 4. To find the radius of curvature of the neutral line, we have

$$\text{rad. curv.} = \rho + n = \frac{Em}{f \sin \alpha}\left(\frac{1}{2} n a^2 - \frac{1}{3} a^3\right);$$

and putting for n its value,

$$\text{rad, curv.} = \frac{Em}{f \sin \alpha} \cdot \frac{a^4}{6(4a + 6h)} = \frac{Ema^4}{36 f \left(h + \frac{2a}{3}\right) \sin \alpha}.$$

If we take a point distant from P by $\frac{2}{3} PQ$, and from this point draw a perpendicular on the line of direction of the force; if this perpendicular $= k$,

$$k = \left(h + \frac{2a}{3}\right) \sin \alpha; \quad \text{rad. curv.} = \rho + n = \frac{Ema^4}{36 fk};$$

or if b be the base of the triangle, $ma = b$, $\rho + n = \dfrac{Ea^3 b}{36 fk}$.

COR. 5. If f be the weight of a length F of the prism, $f = \frac{1}{2} Fab$;

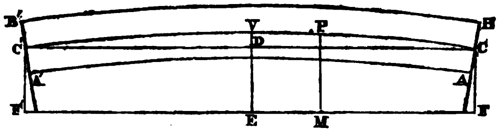
$$\therefore \rho + n = \frac{Ea^2}{18 Fk}.$$

In the same manner we might find the neutral point for prismatic beams of other figures. And the deflexion when they are acted on by given weights would be found in the same manner as before.

Also if the beams are not prismatic, a will be variable; and by putting for it the expression belonging to each case, we may find the deflexion in beams of other forms.

137. PROP. *A rectangular prismatic beam is compressed by a given force acting in a direction parallel to the axis: to find the (small) deflexion.*

Let $ABA'B'$, be the beam, FF' the line in which the force F acts. P any point in the axis. And since the



deflexion is supposed to be small, PM, which is perpendicular to FF', may be considered as perpendicular also to the axis. Hence if a be half the thickness of the beam $(=\frac{1}{2}AB)$ and n the distance of the neutral point above P, $EM = x$, $PM = y$, we have, by Art. 132, $y = \dfrac{a^2}{3n}$, $n = \dfrac{a^2}{3y}$.

Also if ρ be the radius of curvature of the axis CP, by Cor. 2, of the same Article,

$$\rho + n = \frac{E}{F}\cdot\frac{a^2}{3y}; \quad \therefore \rho = \left\{\frac{E}{F} - 1\right\}\frac{a}{3y} = \frac{c^2}{y}, \text{ suppose.}$$

Now $\dfrac{1}{\rho} = -\dfrac{d^2y}{dx^2}$ nearly, because the deflexion is small;

$$\therefore \frac{d^2y}{dx^2} = -\frac{y}{c^2}.$$

$$\text{Integrate, } \therefore \frac{dy^2}{dx^2} = C - \frac{y^2}{c^2}.$$

And if k be EV, the greatest ordinate, $y = k$ when $\dfrac{dy}{dx} = 0$;

$$\therefore \frac{dy^2}{dx^2} = \frac{k^2 - y^2}{c^2}; \quad \frac{1}{\sqrt{(k^2 - y^2)}} = -\frac{1}{c}\frac{dx}{dy};$$

$$\therefore \text{arc}\left(\cos = \frac{y}{k}\right) = \frac{x}{c}; \; x \text{ being measured from } E,$$

$$y = k\cos\frac{x}{c}.$$

Let $l = EF$ = half the length of the beam. And let $h = CF$, the distance of the force from the axis. Therefore when $x = l$, $y = h = FC$,

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$$h = k\cos\frac{l}{c}; \quad y = h \cdot \frac{\cos\frac{x}{c}}{\cos\frac{l}{c}}; \text{ and when } x = 0,$$

$$y = EV = h\sec\frac{l}{c}. \quad \text{Now } DV \text{ the deflexion} = EV - FC;$$

$$\therefore \text{ deflexion} = h\left\{\sec\frac{l}{c} - 1\right\}.$$

$$\text{But } c^2 = \frac{a^2}{3}\left\{\frac{E}{F} - 1\right\}; \quad \therefore \frac{l}{c} = \frac{l}{a}\frac{\sqrt{3F}}{\sqrt{E-F}}.$$

COR. 1. If E be very large compared with F, we shall have the deflexion

$$= h\left\{\sec\frac{l\sqrt{3F}}{a\sqrt{E}} - 1\right\}.$$

COR. 2. The radius of curvature at V

$$= \frac{c^2}{k} = \frac{c^2\cos\frac{l}{c}}{h} = \frac{a^2\cos\frac{l}{c}}{3h}\left\{\frac{E}{F} - 1\right\}.$$

And when E is very large compared with F,

$$\text{rad. curv. at } V = \frac{Ea^2}{3Fh}\cos\frac{l\sqrt{3F}}{a\sqrt{E}}.$$

COR. 3. The deflexion will be greater, as the secant, in Cor. 1, is greater; and when the secant is infinite, the formula will fail. In this case the prism will either be crushed, or will bend so much that the above reasoning is no longer applicable. And this will be the case if the arc be a quadrant. Hence in order that the prism may support a weight with a small deflexion, the weight acting on one side of the axis and parallel to it, we must have

$$\frac{l\sqrt{3F}}{a\sqrt{E}} < \frac{\pi}{2}.$$

$$\frac{l^2}{a^2} < \frac{\pi^2 E}{12F}.$$

Cor. 4. If the force act at the extremities of the axis, $h = 0$; and there will be no deviation except the secant of the arc be infinite; that is, except

$$\frac{l^2}{a^2} = \frac{\pi^2 E}{12 F} = .8225 \frac{E}{F}.$$

Hence we may find the weights which upright columns of given materials will support. Thus, if in fir-wood the modulus E be 10000000 feet, a bar an inch square and 10 feet long will not begin to bend except

$$F = .8225 \times \left(\frac{1}{120}\right)^2 \times 10000000 = 571 \text{ feet};$$

that is, it will bend when pressed by the weight of 571 feet of the same bar, or about 120 pounds, neglecting the pressure arising from the weight of the bar itself.

The modulus of elasticity for iron or steel is about 9000000 feet; for wood, from 4000000 to 10000000; and for stone, probably about 5000000.

Cor. 5. In the same manner we might find the deflexion of a triangular prismatic beam acted on by a longitudinal force. For in this case, supposing E large with respect to F,

$$\rho = \frac{Ea^2}{18 F y}.$$

The preceding Chapters of this work belong to Statics. Those which follow belong to Dynamics. In the Elementary Treatise on Mechanics, two kinds of force were spoken of, *Accelerating Force*, and *Moving Force;* we are now principally concerned with a third kind of force, *Labouring Force.*

CHAPTER VIII.

OF VIS VIVA.

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138. Def. The *vis viva* of a body in motion is a force which is measured by the quantity of matter and the square of the velocity jointly. The vis viva is *equal* to the quantity of matter multiplied into the square of the velocity, the quantities being all represented by numbers.

The vis viva of any system is the sum of the vis viva of each particle.

139. Def. The *Impressed Forces* are the forces which really act upon a body, and give it a certain motion in virtue of the connection of its parts: the *Effective Forces* are the forces which, if the particles of the body were disconnected from each other, must necessarily act, in order that each particle should have at each instant the velocity which it really has.

Thus when a heavy rigid body falls in any manner, the impressed forces are the weights of the different particles of the body: the effective forces are the forces which we find by considering the acceleration of each particle and its mass.

140. Principle. *When motion is communicated to a system, the impressed and the effective forces are equivalent to each other, according to the statical conditions of the system.*

This is an extension of the Principle proved Mech. 104. that the moving force is proportional to the pressure.

In the case of a single particle, directly moved by pressure, the pressure is the impressed force, and the moving force is the effective force; for the moving force is measured by the product of the quantity of matter and of the accelerative force, collected from the actual acceleration. (Mech. 100.)

The principle now before us asserts that the impressed and effective forces, which have been already shewn to be *equal* in the case of direct action, (Mech. 105,) are also *equivalent* in the case of indirect action; as for instance, when weight, or any other pressure, produces motion in a body by levers, sliding contacts, cords, or the like. In all these cases the forces impressed and effective are equivalent, according to the statical conditions of the system.

This principle is an interpretation of the axiom, that Reaction is equal and opposite to Action. The action and reaction in this case, namely, the external forces exerted upon the system to communicate motion and the inertia of the particles to resist the communication of motion, are opposed to each other by means of the system, and act upon it in the manner of opposite pressures. For in this way alone can the mechanical conditions of the system as a combination of material parts be brought into play. Hence these opposite pressures must balance each other on the system, that is, they must be statically equivalent. But the inertia of the particles to resist the communication of motion, is, for each particle, equal to the effective moving force as shown by the actual acceleration. Hence the external impressed forces, taken altogether, must be statically equivalent to the effective moving forces. Q. E. D.

141. This principle was expressed by D'Alembert in another form; under which it is, accordingly, called D'Alembert's Principle: namely under the following form.

When motion is communicated to a system, the quantities of motion gained and lost by the different particles balance each other according to the statical conditions of the system.

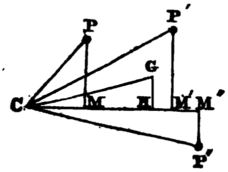
Def. The quantity of motion *lost* by any particle is the quantity by which its momentum is less than it would have been if the particle had been unconnected and acted upon by the same forces during the same time. The quantity of motion *gained* by any particle is, in like manner, the quantity by which its momentum is greater than it would have been if the particle had been unconnected.

By last Article, at every moment the impressed forces of all the particles are equivalent to the effective forces, both sets being moving forces. But the momentum generated in any instant is as the moving force. Therefore the whole momentum which the impressed forces *would* generate in any instant, is equivalent (statically, momentum being the measure of moving force or pressure,) to the whole momentum which the effective forces *do* generate. Hence the excesses of the momentum which the impressed forces would generate above the momentum which the effective forces do generate, in one part of the system, are balanced by the excesses of the momentum which the effective forces do generate over the momentum which the impressed forces would generate, in another part of the system; that is, according to the definition just given, the momentum or quantity of motion lost in one part of the system statically balances the quantity of motion gained in another part of the system. Q. E. D.

142. Prop. *A rigid system consisting of any number of heavy bodies is moveable about a fixt horizontal axis: to find the effective accelerating force on each body.*

Let C be the axis, P, P', P'' the bodies, $CP = a$, $CP' = a'$, $CP'' = a''$: PM, $P'M'$, $P''M''$, perpendiculars on

the horizontal line CM. If P, P', P'' represent the weights of the bodies, they represent also the impressed forces; and the total moment of the impressed forces about C is



$$P.CM + P'.CM' + P''.CM''.$$

Let f be the effective accelerating force which belongs to the point P, and which is of course, perpendicular to CP. Since the whole system turns about the axis with one common angular velocity, the linear velocities of the bodies P, P', P'' are, at the beginning and at each instant, proportional to the distances from the axis, CP, CP', CP''; and therefore the accelerations in each instant are in the same proportion, and therefore the accelerative forces in the same proportion. Hence we have for the accelerative forces belonging to P' and to P'' respectively,

$$f\frac{CP'}{CP} \text{ and } f\frac{CP''}{CP}.$$

Also (Mech. 112) the quantities of matter of P, P', P'',

$$\text{are } \frac{P}{g}, \frac{P'}{g}, \frac{P''}{g} \text{ respectively;}$$

Hence the effective moving forces of P, P', P'', are

$$\frac{f.P}{g}, \frac{f.P'.CP'}{g.CP}, \frac{f.P''.CP''}{g.CP}.$$

And since these forces act perpendicularly at arms CP, CP', CP'', the total moment of the effective forces about C is

$$\frac{f.P.CP}{g} + \frac{f.P'.CP'^2}{g.CP} + \frac{f.P''.CP''^2}{g.CP}.$$

Hence, (Art. 140) equating the moments of the impressed and of the effective forces about C, we have

$$\frac{f(P.CP^2 + P'.CP'^2 + P''.CP''^2)}{g.CP}$$
$$= P.CM + P'.CM' + P''.CM''.$$

Hence we have

$$f = \frac{(P.CM + P'.CM' + P''.CM'')\,g.CP}{P.CP^2 + P'.CP'^2 + P''.CP''^2}.$$

Cor. 1. If the symbol $\Sigma.P.CM$, or $\Sigma.P.CP^2$ be used to represent "the sum of all the quantities" which differ only by substituting the bodies P, P', P'', for one another; we have

$$f = \frac{g.CP.\Sigma.P.CM}{\Sigma.P.CM^2}.$$

Cor. 2. The expression would manifestly be the same if there were any number of bodies. Hence in the case of any number of bodies

$$f = \frac{g.CP.\Sigma.P.CM}{\Sigma.P.CP^2}.$$

Cor. 3. If G be the center of gravity of the system, and if the vertical line GH meet the horizontal line CH, (Mech. 43)

$$\Sigma.P.CM = CH\,(P + P' + P'' + \&c.) = CH.\Sigma.P.$$

Hence, if $CGH = \theta$, $\Sigma.P.CM = CG \sin\theta.\Sigma P$,

$$f = \frac{g.CG \sin\theta.CP.\Sigma.P}{\Sigma.P.CP^2},$$

the quantity $P.CP^2 + P'.CP'^2 + P''.CP''^2$, or $\Sigma.P.CP^2$ is called the *rotatory inertia* of the system; sometimes, the *inertia of rotation*. It may be found in given systems by the Integral Calculus.

143. Prop. *A rigid system moves about a horizontal axis by the force of gravity: the vis viva acquired by the system at any time from rest is the same as it would have been if the particles, unconnected, had fallen through the same vertical spaces from rest.*

Let Y be the first distance of the center of gravity below the horizontal line CH; and let H be the vertical space

through which the center of gravity descends from rest. And let y, y', y'' be the first distances of P, P', P'' below the horizontal line CH; and $y+h$, $y'+h'$, $y''+h''$ the distances of P, P', P'' below the same horizontal line when G has descended through H. Then, by the property of the center of gravity, (Mech. 43)

$$(P+P'+P'')\,Y = Py + P'y' + P''y''$$

$$(P+P'+P'')(Y+H) = P(y+h) + P'(y'+h') + P''(y''+h'').$$

Hence subtracting

$$(P+P'+P'')\,H = Ph + P'h' + P''h''.$$

But if P, P', P'' has descended freely through the same vertical spaces (whether their paths were vertical or not), and if u, u', u'' were the velocities required by these descents, we should have

$$u^2 = 2gh,\ \ u'^2 = 2gh',\ \ u''^2 = 2gh'';$$

Hence

$$(P+P'+P'')\,2gH = Pu^2 + P'u'^2 + P''u''^2.$$

Now the effective accelerating force which acts on the center of gravity, by Cor. 3 of the last Article, putting R for CG, is

$$f = \frac{gR^2 . \Sigma . P}{\Sigma . P . CP^2} \sin\theta,$$

which is the same as if a single point moving at the extremity of a radius R, were acted upon by a vertical force

$$G = \frac{gR^2 \Sigma . P}{\Sigma . P . CP^2}.$$

But in that case we should have $V^2 = 2GH$. (Mech. 120.) Hence we have, V being the actual velocity of G,

$$V^2 = \frac{2gR^2 \Sigma . P}{\Sigma . P . CP^2} H,$$

$$\frac{V^2}{R^2} \Sigma . P . CP^2 = 2gH . \Sigma . P.$$

But if v, v', v'' be the actual velocities of P, P', P'', we have $v = V.\frac{CP}{R}$, because the angular velocity is the same for all; and so of P', P''. Hence, interpreting Σ,

$$\frac{V^2}{R^2}\Sigma.P.CP^2 = \frac{V^2}{R^2}(P.CP^2 + P'.CP'^2 + P''.CP''^2)$$

$$= Pv^2 + P'v'^2 + P''v''^2.$$

Whence $Pv^2 + P'v'^2 + P''v''^2 = 2gH(P + P' + P'')$,

by what precedes $= (P + P' + P'')u^2$,

or $Pv^2 + P'v'^2 + P''v''^2 = Pu^2 + P'u'^2 + P''u''^2$,

which is the Proposition to be proved. And it is evident that the same proof would apply to any number of bodies. Hence, for any rigid system,

$$\Sigma.Pv^2 = \Sigma.Pu^2.$$

Cor. $\Sigma.Pv^2 = \Sigma.P.2gh.$

144. Prop. *A rigid system moves about a fixt horizontal axis by the force of gravity: the increase of vis viva of the system in moving from one position to another is equal to the whole vis viva which all the particles would have acquired if they had fallen unconnected through the same vertical spaces.*

Let v_1, v_2 be the velocities of any particle P in the first and in the second position; and in like manner, v_1' and v_2' the velocities of P'; and so on. Let h_1 be the vertical space through which P has fallen from rest to the first position, and h_2 the vertical space from rest to the second position; and let h_1', h_2' be the same quantities for P'; and so on. Then, by the Cor. to last Proposition, we have

$$Pv_1^2 + P'v_1'^2 + \&c. = P.2gh_1 + P'.2gh' + \&c.$$

$$Pv_2^2 + P'v_2'^2 + \&c. = P.2gh_2 + P'.2gh_2' + \&c.$$

Hence subtracting, $P(v_2^2 - v_1^2) + P'(v_2'^2 - v_1'^2) + \&c.$

$$= P.2g(h_2 - h_1) + P'.2g(h_2' - h_1') + \&c.$$

But if u, u', &c. be the velocities acquired by P, P', &c. in falling unconnected from the first to the second position, we have

$$u^2 = 2g(h_2 - h_1),\ u'^2 = 2g(h_2' - h_1'),\ \&c.$$

And the second side of the above equation becomes

$$Pu^2 + P'u'^2 + \&c.$$

which (Art. 138) is the *vis viva* acquired by the particles unconnected. Hence

$$\Sigma . P(v_2^2 - v_1^2) = \Sigma . Pu^2.$$

Cor. $\Sigma P(v_2^2 - v_1^2) = 2g\Sigma . P(h_2 - h_1)$.

145. Prop. *If a particle* P, *acted upon by a constant force* Pf, *move in any manner, through a space, which (estimated in the direction of the force* Pf) *is* s, *the increase of vis viva during this motion is equal to the vis viva which would have been communicated to the particle* P *moving from rest, by the action of the same force through the same space.*

The force Pf is an accelerating force, f, acting upon a quantity of matter P. Let v_1 be P's velocity at the beginning, and v_2 the velocity at the end of the motion now considered. And let r be the space through which the accelerating force f must have acted to generate the velocity v_1.

Hence

$$v_1^2 = 2fr,\ v_2^2 = 2f(r + s) = 2fr + 2fs = v_1^2 + 2fs.$$

Hence $v_2^2 - v_1^2 = 2fs$; or if u be the velocity which would have been generated by an accelerating force f, or a moving force Pf, acting on P through a space s, from rest,

$$v_2^2 - v_1^2 = u^2.$$

Hence $Pv_2^2 - Pv_1^2 = Pu^2$,

which is the Proposition.

Cor. Hence the vis viva, generated in a particle by a constant force acting through any space, is independent of the previous motion of the particle.

146. Prop. *Any system whatever being moved by any forces, the increase of vis viva of the system in moving from one position to another, is equal to the whole vis viva which all the particles acted upon by the forces would have acquired, if each particle had been drawn, unconnected, through the same space (estimated in the direction of the force which acts upon it) as it moves through in the actual motion of the system.*

Let Pf, $P'f'$, &c. be the Impressed Forces which act upon the particles P, P', &c.; and let Q, Q', &c. be the quantities of matter of the particles moved. In any instant of time, let the system move from one position to another, all the parts moving through small spaces; and let ϕ, ϕ', &c. be the velocities of the particles P, P', &c. in the direction of the forces Pf, $P'f'$ &c.: v, v', &c. the velocities of the particles Q, Q', &c. in the actual direction of their motions. And if t be the time, the effective accelerating forces belonging to the particles Q, Q', &c. are respectively $\frac{dv}{dt}$, $\frac{dv}{dt}$, &c., and the effective moving forces are $Q\frac{dv}{dt}$, $Q'\frac{dv'}{dt}$, &c.; and the Impressed and Effective Forces balance each other statically; therefore (Art. 45) the principle of virtual velocities is applicable to them: and ϕ, ϕ' &c., v, v', &c. are the virtual velocities.

$$\text{Therefore, } Qv\frac{dv}{dt} + Qv'\frac{dv'}{dt} + \text{\&c.} = Pf\phi + P'f'\phi' + \text{\&c.}$$

But if s be a line drawn to P in the direction of the force Pf, and measured from a fixt point, we have $\frac{ds}{dt} = \phi$; and in like manner, $\frac{ds'}{dt} = \phi'$; and so on.

$$\text{Hence } Qv\frac{dv}{dt} + Q'v'\frac{dv'}{dt} + \&c.$$

$$= Pf\frac{ds}{dt} + P'f'\frac{ds'}{dt} + \&c.$$

$$\text{Hence } Q.2v\frac{dv}{dt} + Q'.2v'\frac{dv'}{dt} = 2Pf\frac{ds}{dt} + 2P'f'\frac{ds'}{dt} + \&c.$$

And this is the differential, taken with regard to t, of the integral equation $Qv^2 + Q'v'^2 + \&c. = 2Pfs + 2P'f's' + \&c.$; the integral being taken for a small instant, during which the forces Pf, $P'f'$, &c. may be supposed to be constant. Hence if v_1, v_2 be the velocities of Q at the beginning and end of this instant, and s_1 and s_2 the corresponding values of s; and similarly, for Q', P', &c.; we shall have

$$Q(v_2^2 - v_1^2) + Q'(v_2'^2 - v_1'^2) + \&c.$$
$$= 2Pf(s_2 - s_1) + 2P'f'(s_2' - s_1') + \&c.$$

But if u, u', &c. be, as in last Article, the velocities which the forces Pf, $P'f'$, &c. would generate, in the spaces $s_2 - s_1$, $s_2' - s_1'$, &c.; $u^2 = 2f(s_2 - s_1)$, $u'^2 = 2f'(s_2' - s_1')$, &c. Hence $Q(v_2^2 - v_1^2) + Q'(v_2'^2 - v_1'^2) + \&c. = Pu^2 + P'u'^2 + \&c.$

Hence the vis viva generated in *any small instant* is equal to that which would have been generated through the same space, if the particles had been unconnected. Hence, this is the case in a second instant: and therefore the vis viva generated in two successive instants is equal to that which would have been generated in two successive instants through the same space if the particles had been unconnected. And the same reasoning applies to any number of instants. Hence the vis viva generated in any interval is equal to the vis viva which would have been generated if the particles, unconnected, had been acted upon by the same forces, through the same spaces. Q. E. D.

This Proposition is called the Principle of the *Conservation of Vis Viva*. It appears from the demonstration of this Principle, that the vis viva generated or added during any interval in the motion of any material system does not depend upon the velocities or directions of the motions of the particles at the beginning of the interval, nor upon the paths described by the particles, but only upon the Impressed Forces and the distances described by the particles in the direction of those forces. Whatever be the construction of the machine, the vis viva due to these forces and distances is preserved without loss.

This is not true if there be *abrupt* changes of velocity, as will be seen hereafter.

147. If the Impressed Forces be the weight of heavy bodies, the Principle of the Conservation of vis viva becomes this Proposition.

When any weights act upon a machine in motion, the vis viva generated in the machine in any interval is equal to that which the weights would have acquired by falling freely through the same vertical distances.

If weights are made to ascend by the machine, there is a vis viva destroyed, corresponding to the vertical ascent of those weights.

148. PROP. *When any system moves from rest by the action of gravity, the actual descent of the center of gravity is equal to its potential ascent.*

The *potential ascent* is the height through which the center of gravity would ascend, if each particle were to be disconnected from the system, and projected vertically upwards with its actual velocity.

Let h, h' &c. be the actual descent of the particles P, P', &c. from rest; v, v' the actual velocities. Then (Art 147,)

$$Pv^2 + P'v'^2 + \&c. = 2gPh + 2gP'h' + \&c.$$

Let k be the height to which P would ascend if projected vertically upwards with a velocity v; k' the same for v'; and so on.

$$\text{Hence } v^2 = 2gk, \; v'^2 = 2gk', \; \&c.$$

$$\text{and } 2gPk + 2gP'k' + \&c. = 2gPh + 2gP'h' + \&c.$$

$$\text{whence } Pk + P'k' + \&c. = Ph + P'h' + \&c.$$

But if y, y', &c. be the distances of P, P', &c. above a horizontal plane at the instant of disconnection, the original heights were $y + h$, $y' + h'$, &c. And the original height of the center of gravity was

$$\frac{P(y+h) + P'(y'+h') + \&c.}{P + P' + \&c.},$$

$$\text{which is } \frac{Py + P'y' + \&c.}{P + P' + \&c.} + \frac{Ph + P'h' + \&c.}{P + P' + \&c.};$$

of which the first fraction is the height of the center of gravity above the plane at the moment of disconnection, and the second fraction is the actual descent of the center. Also in the same manner the potential ascent is

$$\frac{Pk + P'k' + \&c.}{P + P' - \&c.}.$$

And this is equal to the actual descent, because their numerators have been proved equal. Therefore, &c. Q. E. D.

COR. When a machine or system is in motion, the actual height to which the center of gravity will ascend in virtue of the velocities of the parts and their connexion, will not be altered, if at any moment the connexion of the parts be changed in any manner.

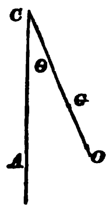
For both before and after the change, the height to which the center of gravity will ascend will be the actual

descent by which the parts acquired their velocities, the motion in each case being reversed. And in each case, the actual descent will be equal to the potential ascent, the parts being disconnected at the given moment. Therefore the actual descent is the same in the two cases, and is not altered by the change in the connexion. And therefore the actual ascent is not altered.

149. Prop. *To find the center of oscillation of any rigid system moveable about a fixt horizontal axis.*

The center of oscillation is a point in a rotatory system, in which if a particle were placed, it would oscillate by the force of gravity with exactly the same velocity as the system itself.

Let C be the axis, G, the center of gravity, O, a point in CG. Then by Art. 142, Cor. 3, if f be the accelerating force upon O, θ the angle which CG makes with a vertical line CA in the direction of its motion; we have



$$f = \frac{g \, . \, CG \sin\theta \, . \, CO \, . \, \Sigma \, . \, P}{\Sigma \, . \, P \, . \, CP^2}.$$

But if we have a heavy particle placed at O, the accelerating force upon it in the direction of its motion is $g \sin\theta$. Now if the accelerating force upon a particle at O be at every instant equal to the accelerating force which acts upon O in the actual state of the system, the motion of O will be the same on the two suppositions, and O will be the center of oscillation. Hence O will be the center of oscillation if

$$g \sin\theta = \frac{g \, . \, CG \sin\theta \, . \, CO \, . \, \Sigma \, . \, P}{\Sigma \, . \, P \, . \, CP^2};$$

$$\text{whence } CO = \frac{\Sigma \, . \, P \, . \, CP^2}{CG \, . \, \Sigma \, . \, P}.$$

CHAPTER IX.

OF THE MEASURE OF LABOURING FORCE.

150. In the cases which come under the consideration of the engineer, force is exerted by machines, and motion is modified by mechanism, with a view to *do work*. This work consists in shaping or moving certain portions of matter. Thus, we have to grind bodies, to polish them, to divide them into parts, to twist or weave together threads or cords, to draw a carriage along a road, to bend springs, to move stones or masses of wood or metal. In these cases we have to overcome the cohesion of matter, inertia, elasticity, weight. All such processes may be spoken of as work done by labour. The force which produces these effects, and which is measured by the effect thus produced, will hence be termed *Labouring Force*.

151. Labouring force increases with the pressure exerted at the work, and with the space through which it is exerted. If the work done be, for example, to draw a carriage along a rough road, the work done increases as the resistance offered by the road to locomotion increases; and it increases also as the space moved through on the road increases. If the work done be to raise a weight, the work of raising ten pounds is greater than the work of raising five pounds through the same height; and the work done in raising ten pounds through ten feet of height is greater than the work done in raising ten pounds through five feet of height. To saw through a tree four feet diameter is to do more work

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than to saw through a tree three feet diameter of the same wood; but if the wood be harder, in the latter case the work done is increased by that circumstance.

Thus the work done, and the labouring force as measured by the effect, increases both with the resistance overcome, and with the space through which it is overcome. But it does not follow directly from this, that the labouring force is measured by the product of resistance and space described. For the *moving force* which is requisite to project a heavy body to a certain height, increases with the weight of the body, and with the given height. But it does not increase in proportion to the height. To project a body so that it shall ascend vertically through ten feet requires a force, not *twice*, but *four* times as great as to project it so as to ascend through five feet. If therefore labouring force were the same thing as moving force, it would not be proportional to the space described. But, in fact, labouring force is proportional to the resistance and the space jointly, as we shall show.

152. *Labouring force is measured by the product of the resistance overcome, and the space through which it is overcome.*

For in doing work, the resistance is supposed to spring up afresh in every new particle of the space. When a body is projected upwards, the first impulse being given, the body ascends a certain space by the law of motion, without any continued application of force: but if we raise a body slowly by a rope, we cannot for an instant relax our exertion; a new pressure is requisite to draw the body through each successive particle of space.

This latter is the case of labouring force. In like manner, in dragging a carriage, in grinding, cutting, sawing,

polishing, twisting, a constant exertion of force upon the object operated on is requisite in order to overcome the constant resistance. If it require a certain labouring force to saw through one inch of a uniform block, it must require an equal labouring force to saw through the next inch of the same block. After each exertion of the force, the object will remain where it is and as it is, except a new exertion of force carry on the work. And thus the labouring force exerted is proportional to the space through which the resistance is overcome.

Also the effect increases proportionally to the resistance. To saw through two equal trees is twice as much work as to saw through one, whether the work be done simultaneously or successively. Hence the labouring force is proportional to the resistance. Therefore it is proportional to the resistance and the space jointly; and may be measured by the product of two numbers representing these quantities.

Cor. 1. If f be the resistance overcome, s the space through which it is overcome, then f being constant, the labouring force is $f \times s$.

Cor. 2. If f be variable, δs an element of space through which f may be conceived to be constant; $f\delta s$ is the element of the labouring force for the space δs.

Cor. 3. The labouring force for a finite space is to be found by taking the sum of all the elements $f\delta s$. This may be done by the integral calculus; and, f being a function of s, the labouring force will be $\int f ds$, or in other notations $\int_s f$.

153. *Labouring force may be measured by a given weight raised through a given vertical space.*

For any resistance may be expressed by weight; and the overcoming the resistance may be represented by raising the weight through a vertical space: it being always supposed that the weight will remain at the point to which it is thus raised. Thus if it requires a force of one hundred pounds to draw a carriage along a road, the labouring force employed in drawing this carriage twenty feet may be measured by the labouring force which would raise a weight of one hundred pounds through twenty feet.

Hence the labouring force is independent of the nature of the work done. The nature of the work done may be infinitely modified by the mechanism employed; but the labouring force expended cannot, as we shall see, be increased or diminished by any of these transformations. When a certain quantity of work is to be done, the labouring force which is requisite is the same, whatever be the mechanism by which it is done, except so far as the mechanism is bad, and consumes labouring force uselessly. And when we have a certain quantity of labouring force at our command, it may be employed in doing a corresponding quantity of any kind of work whatever, provided we are skilful enough to devise mechanism which is fit to do the work.

154. *Labouring Force is the labour that we pay for.* In many cases the work to be done may be performed by various agencies; by men, by horses, by water, by wind, by steam. In these cases, that is the cheapest mode of doing the work which gives us the requisite labouring force at the smallest expense: and the price men are willing to pay, and customarily do pay, is proportional to the quantity of labouring force which they purchase. Labouring force enters into the prices of articles produced by man; as the wages of labour, so far as the labouring force is executed directly by man; as the reward for capital, so far as the

labouring force arises from accumulated capital. But wages of labour are paid, not only for man's labour, but for labouring force when arising from machinery. In our towns in which large manufactories exist, such establishments often generate by their machinery more labouring force than they need; and the surplus (transferred by an axis, or in some other way, to a distance) is hired by other persons, and employed for the purposes of the most various kinds of work. In such towns, we often read advertisements of "power to be disposed of to a large amount." The *power* here spoken of is labouring force. The cost is proportional to the quantity of labouring force so bought and sold.

155. Various denominations have been applied to this Labouring Force by different writers. Smeaton called it *mechanical power;* Carnot, *moment d'activité;* Coulomb, Navier, and others, *quantity of action.* Mr. Davies Gilbert proposed to term *efficiency* what we call labouring force. In certain kinds of work, our engineers use the term *duty* (the *duty* of a Cornish steam-engine) to express the quantity of labouring force as it appears in the work done, (which is principally the draining of mines.) But M. Poncelet, and other French engineers, who have employed this conception of labouring force most instructively in their books, call it *travail;* and this term is now very generally diffused and adopted. The term *Labouring Force,* which we use in the present work, has the advantage of marking the distinction between the force measured by the work done, and other acceptations of force in which it is measured by other kinds of effect, as moving force, accelerating force, statical force. The term *Labouring Force* has also the advantage of corresponding with sufficient closeness to the French term *Travail.*

156. ***The Unit of Labouring Force is called the Dynamical Unit, and is the Labouring Force requisite to raise one pound through one foot.***

By what has been said, it appears that we may take as the unit of labouring force, a given weight raised through a given space. Many different units thus determined have been proposed by different writers. The French engineers, Mongolfier, Hachette, Clement, &c. took as the unit 1 cube metre of water, or 1000 kilogrammes of weight, raised 1 metre; others, as M. Dupin, have proposed to adopt 1000 cube metres or *tonneaux* of water raised one metre. It was proposed to term these units *Dynamies*, or *Dynames*.

The unit now in use among French engineers is 1 kilogramme raised 1 metre, which is written $1^{\text{k.m.}}$ or 1^{km}, and called a *kilogrammetre*.

A dynamical unit much used in measuring the labouring force of steam-engines is a *horsepower*. The actual labouring force of a horse is a variable quantity, but we may hypothetically suppose it to be constant. By Bolton and Watt a horsepower was taken to be 33,000 pounds raised 1 foot in 1 minute; and this may be still assumed as the hypothetical horsepower with reference to steam-engines. It is sometimes called by French writers *cheval vapeur*.

The standard mentioned at the head of this article, one pound raised one foot, is the most general and convenient measure, and will be taken in this work as the *dynamical unit*. We may write this phrase, for the sake of abbreviation, *dynam*. And if we please, we may also read it so. Thus we should say that a *hypothetical horsepower* is 33,000 *dynams per minute;* or 550 dynams per second.

The estimate of a horsepower used by French engineers is 75^{km} per second. As the metre is 3.28 feet, and the

kilogramme 2.2 pounds, the *kilogrammetre* is 7.216 *dynams*, and the *cheval vapeur* is 525 dynams per second; which agrees nearly with the English estimate.

We now proceed to prove that the amount of labouring force, as shewn in its effects, is not altered by the mechanism through which it acts, the kind of motion which it produces, or the rate at which the work is done.

157. PROP. *The amount of Labouring Force is not altered by any change in the Mechanism through which it acts.*

Let F be the pressure exerted upon the first piece of the mechanism in the direction of its motion, and S the space through which it moves in any time; let f be the pressure exerted by the last piece of the mechanism upon the work, and s the space through which this piece moves. Then whatever be the train of mechanism, we have, by the Principle of Virtual Velocities, $f \times s = F \times S$. But $f \times s$ is the measure of the labouring force employed upon the work during this time. And since this is always $= F \times S$, it is the same whatever be the mechanism by which the work is done.

COR. By the same reasoning the amount of labouring force corresponding to a given space, is not altered by altering the velocity of working.

In this reasoning F and f are pressures, and are supposed not to be modified by the velocity with which the work is done. But in many cases the pressure exerted upon a given piece of a machine changes according to the velocity with which the piece is moved. In such cases the labouring force may be altered by altering the rate of working; and there may be a rate of working for which the labouring force is a *maximum*.

158. The pressure exerted at the work must always be greater than the greatest resistance which is to be overcome; otherwise the work could not go on. In order to economize force, the pressure employed is usually not much greater than is requisite to overcome the greatest resistance. Hence, if the greatest resistance be much greater than the mean resistance, the pressure employed must be much greater than the pressure which would be requisite, if, the mean resistance remaining the same, we could remove or much diminish the greatest resistance.

Thus in drawing a carriage, which one horse could draw on level ground, it may be requisite to harness four in order to ascend the hills which occur. The force requisite to draw a carriage along a road varies with the nature of the road. The requisite force of traction varies according to the friction; and is a certain constant fraction of the weight of the carriage and load. For a given road, let this fraction be $\frac{1}{20}$; and let there occur in the road a hill of which the slope is 1 in 10. Then, by the property of the inclined plane, a force of $\frac{1}{10}$ the load is requisite to draw the carriage up the hill, besides what is requisite to overcome the friction. Hence the requisite force of traction is $\frac{1}{20}$ on the level, and $\frac{1}{20}+\frac{1}{10}$, or $\frac{3}{20}$ on the slope. If one horse be sufficient in the first case, three will be requisite in the other.

159. Hence if the friction of roads be diminished, the maximum inclination of the parts of the road must be diminished proportionally, in order that the power of traction requisite for travelling may be diminished in a corresponding degree.

Thus in the case just mentioned, if the road be improved so that the friction becomes $\frac{1}{30}$, the slopes remaining unchanged, the forces of traction requisite on the level and the slope are $\frac{1}{30}$ and $\frac{1}{30}+\frac{1}{10}=\frac{4}{30}$: and if, as before, the force of a horse be $\frac{1}{20}$, three horses will still be requisite to ascend the hills. But if the hills be cut down so that the ascent is only 1 in 60, the greatest force requisite will be $\frac{1}{30}+\frac{1}{60}=\frac{1}{20}$; and a single horse will suffice, even for ascending the hills.

If the road were to become a rail-road with a friction of only $\frac{1}{240}$ the load, and if the slopes were as much as 1 in 30, we could not ascend these slopes without a force 9 times as great as that which we want on the level parts. But if in these cases we can reduce the slope very much, for instance to 1 in 300, the greatest force then requisite is

$$\frac{1}{340}+\frac{1}{300}=\frac{3}{400}=\frac{1}{133} \text{ of the load.}$$

160. Prop. *When any pressures* f, f′, f″, *act upon any machine, the labouring force is proportional to the vis viva which these forces would have generated, acting through the same spaces.*

The pressure f may be constant or variable. First let f be constant. If a constant pressure f act upon a given body m, the accelerating force is $\frac{f}{m}$ (Mech. 112). And if this force acting through a space s, produce a velocity v, we have $v^2=2\frac{f}{m}s$; and $mv^2=2fs$.

Now mv^2 is the vis viva generated by the force f, in acting through the space s. And it appears from what has been said that mv^2 is proportional to fs.

In like manner, if any other forces f', f'', &c. were to act on the machine through spaces s', s'', &c. and if v', v'', &c. were velocities which these forces would generate in bodies m', m'', &c.; we should have

$$m'v'^2 = 2f's'; \quad m''v''^2 = 2f''s''; \text{ \&c.}$$

Hence we have

$$\tfrac{1}{2}(mv^2 + mv'^2 + m''v''^2) = fs + f's' + f''s''.$$

Or the whole vis viva is proportional to the whole labouring force.

Next, let f be variable. Let δs be a small space through which f is conceived to be constant. Then $f\delta s$ is the corresponding element of the labouring force. But if f act upon a body m through a space δs, and augment its velocity from v to $v + \delta v$, we have (Mech. 78),

$$\frac{2f}{m}\delta s = (v + \delta v)^2 - v^2 = 2v\delta v + (\delta v)^2.$$

Hence taking the limit, when $\dfrac{\delta v}{ds}$ becomes $\dfrac{dv}{ds}$,

$$mv\frac{dv}{ds} = 2f; \quad \text{whence } \tfrac{1}{2}mv^2 = \textstyle\int_s f.$$

But $\int_s f$, (or $\int f ds$ in other notations) is the sum of all the ultimate elements, $f\delta s$, of the labouring force; and mv^2 is the vis viva.

And in like manner we should have

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}m'v'^2 + \tfrac{1}{2}m''v''^2 = \textstyle\int_s f + \int_{s'} f' + \int_{s''} f''.$$

Cor. If we take the same measures on the two sides of the equation, *the labouring force is half the vis viva.* On this supposition, f is considered as a weight, and may

be measured in pounds; m is quantity of matter, and if w be the weight of m, $m = \frac{w}{g}$, (Mech. 112).

161. Prop. *When any pressures* f, f′, f″ *act through given spaces, the total labouring force is the same, whether they act jointly upon one or more machines, or produce their effects separately and independently.*

For by Art. 146, the total vis viva generated is the same, whether the forces act upon any system whatever, or act upon any masses separately and independently. And since the labouring force is as the vis viva, the labouring force is the same on the two suppositions.

Cor. 1. Hence it appears, as was stated above, that the total labouring force is not increased or diminished by any change in the mode of action.

Cor. 2. For instance, no labouring force is lost by the oblique action of a piston-rod on a crank.

We have now to consider how the labouring force is consumed in doing work.

162. The work which is the object of a machine exerts a resistance which must be overcome by the machine. But besides this *useful resistance*, there are other resistances which, though inevitable, impede the work: these may be called *impeding resistances*. They are such as the resistance of the air, friction, the forces producing waste and change of form in parts which we wish to have durable and invariable. These resistances continually exist in machines. If we have a machine consisting of wheels, and employed to grind or polish pieces of metal, it cannot do this without at the same time grinding and polishing its own axles, though this is not a part of the work contemplated. The resistance arising from the latter operation is the impeding resistance, from the former, it is the useful resistance.

163. *The total labouring force is consumed by the useful and the impeding resistances taken together.*

The total labouring force is measured by the resistances, multiplied into the spaces through which they are overcome, including *all* the resistances which act upon the system. And all these resistances which are not part of the work done, that is, of the useful resistance, diminish the work done, and may be reckoned as impeding resistances. Hence the two kinds of resistance taken together account for the whole of the labouring force.

The labouring force may be considered as *consumed;* for it is lost and cannot be recovered after being used. When a body has its velocity diminished by rubbing against an immovable obstacle, the force thus lost cannot again be brought into operation. In like manner in all operations by which the particles of matter are separated or made to cohere, as cutting, sawing, grinding, polishing, moulding, spinning, weaving, the force exerted disappears, and cannot again be brought into play as force. It is consumed. In cases where the force is employed in raising a weight, moving a mass, or bending a spring, the force is stored up; and may be again brought into play and used. But if this future use of the force thus stored up is not contemplated, the force thus employed may also be considered as consumed. The force by which a sack of grain is hoisted to the highest floor of a mill is consumed in the operation; for though the heavy sack, when arrived at the highest point, might be used as a weight to do other work, we know that this will not be done. The result contemplated is that the corn will be ground; and the force employed in raising the sack to the upper floor is a part of the force consumed in the complex operation of grinding.

Thus a portion of the labouring force is consumed in overcoming the useful resistances.

The remainder of the labouring force is consumed in overcoming the impeding resistances. For force employed in overcoming friction, resistance of fluids, and the like, is lost, and cannot again be brought into play.

Thus the whole of the labouring force is consumed by the useful and impeding resistances taken together.

164. Hence in calculating the labouring force requisite for any purpose, we must take into account that part of such force which is requisite to overcome the impeding resistance.

For example, let a machine contain a wheel a ton weight, which turns ten times a minute on an axle a foot diameter, the friction at the axle being $\frac{1}{5}$ the pressure; what is the labouring force required to overcome this impeding resistance?

The friction $= \frac{1}{5}$ pressure $= \frac{1}{5} \times 2240$ pounds $= 448$ pounds. This force acts at the circumference of the axle, and therefore describes in one minute ten circumferences of the axle $= 10 \times \pi$ feet $= 31\frac{3}{7}$ feet nearly. Therefore the labouring force requisite to overcome the impeding resistance at the axle of this wheel is $= 31\frac{3}{7} \times 448$ dynams $= 14{,}080$ dynams per minute $= \frac{3}{7}$horse-power, nearly.

The labouring force requisite for any work may with proper elements be calculated from the work done. The following are examples taken from French writers.

Saw-mill of Volone, (Hautes Alpes).
(*Taffe, Application des Principes de Mécanique*, p. 123).

165. The saw in eleven minutes cut a plank of green oak 3^{m} long and $0^{m}.4$ thick; which makes a surface of 1.2^{m}

superficial. But according to Navier, to saw 1^{m} superficial of green oak gives rise to 43333^{km} of useful resistance. Therefore in eleven minutes the useful resistance overcome was 51666.3^{km}; in one minute it was 4697^{km}; and in one second it was 78^{km}, nearly; or a little more than one-horse power.

This is the amount of the useful resistance overcome; but it appears that in the machine in question labouring force of the amount of 258^{km} was expended. Hence the impeding resistances consumed 180^{km} of this amount.

Fire Engine.

(*Poncelet, Mécanique Industrielle. Application.* Art. 150.)

166. Suppose that we have to throw water out of a fire-engine, with a uniform velocity of 15 *metres* per second, and that there must be thrown upon the object every second a quantity of 6 *litres*, which is 6 *kilogrammes*. Here, the velocity is 15, and the quantity of matter to which this velocity is communicated, $\left(\frac{w}{g}, \text{ see Art. } 160\right)$ is $\frac{6^{k}}{9.81}$.

Hence the vis viva generated in each second is

$$\frac{6^{k} \times (15)^{2}}{9.81} = 137.6 \text{ in kilogrammes and metres.}$$

Hence (Art. 160) the half of this, $68^{km}.8$, is the labouring force requisite to do this work, besides what is requisite to overcome impeding resistances.

CHAPTER X.

OF RESERVOIRS OF LABOURING FORCE.

167. When Labouring Force is employed in any of the most usual kinds of work, as in destroying the cohesion of matter, in overcoming friction, and the like, it is, as we have seen, consumed in the process. But if it be employed in raising heavy matter, in communicating motion to inert masses, or in compressing elastic bodies, it is expended for the moment, but the result still continues in a form in which it may be again called into action as labouring force.

Thus if labouring force be employed in raising heavy bodies into a position from which they can again descend so as to do work by their descent, the labouring force so employed may be brought into play undiminished, by letting the bodies descend, attached to proper mechanism. A weight of 1000 pounds raised to a height of 10 feet, retains the labouring force of 10,000 dynams which was employed in raising it; and can be employed to do work to the amount of 10,000 dynams by its descent through the 10 feet. Labouring force may be employed to drive piles in this way. If a weight of 1000 pounds be raised through 10 feet, and then left to fall through the same 10 feet by gravity, the square of the velocity acquired is ($2gs =$) $20g$; and the mass being $\frac{1000}{g}$, the vis viva of the weight at the bottom of the descent is 20,000; and therefore the labouring force, which is half the vis viva, (Art. 160) is 10,000 dynams.

In like manner, men may exert labouring force by ascending to a high point by the use of their limbs, and then descending as weights in a machine. If the weight of a man be 140 pounds, when he has ascended 100 feet, he has exerted a labouring force of 14,000 dynams; and if he then descend in a basket as counterpoise to a weight to be raised, this labouring force is brought into action.

In these cases the weights, whether of inert matter or of the man himself, placed at a height from which they can descend by the force of gravity, may be considered as *reservoirs* or *stores* of the labouring force which has been employed in raising the weights: and the labouring force thus *reserved* or *stored up* is measured by the product of the weight and the possible descent.

168. The same is the case if the matter thus existing at an elevated point be fluid. A mill-pond contains labouring force equal to the product of the weight of the water, and of the possible descent. It is a reservoir of labouring force; of the force which has raised the water, if it have been raised by mechanical means; of the force supplied by nature, if the water have collected at this height by natural causes. In the latter case, if we prevent the water from flowing down to a lower level without doing work, we make our reservoir of water a reservoir of labouring force.

A mill-pond 2 feet deep, and 100 feet square, with a fall of 8 feet, containing therefore 20,000 cubic feet of water, or (since a cubic foot of water is $62\frac{1}{2}$ pounds) 1,250,000 pounds, is a reservoir of labouring force to the amount of $8 \times 1{,}250{,}000$ or 10,000,000 dynams.

169. In like manner, elastic bodies may be made reservoirs of labouring force; so far, that is, as they are *perfectly* elastic. If force be employed to compress a spring, which is

then detained in a state of compression, the spring by its unbending may bring into exercise the force which was employed in compressing it. Thus the force which we employ for a few seconds in winding up a clock, is stored up in the main spring, and is afterwards consumed in a day, or a week, in giving motion to the wheels of the clock. The force by which the air is compressed in the chamber of an air-gun is preserved there, and is brought into play to communicate motion to the ball; and the vis viva of the ball thus put in motion is, (impeding resistances not being allowed for,) the half of the labouring force thus stored up.

If the bodies thus employed be imperfectly elastic, a part of the force which thus compresses them is irrecoverably consumed in altering their shape or internal condition; as we shall further see in considering the case of percussion.

170. The inertia of bodies, that is, their tendency to preserve the motion which they possess, in conformity with the first law of motion, is also a reservoir of force. Thus if a labouring force have been employed in giving to a mass of 1000 pounds a velocity of 10 feet in a second, this mass (if so sustained that its force is not consumed by resistances) contains the labouring force which has given to it its motion, and may be made to give out this labouring force so as to do work. In the case just mentioned, the vis viva of the mass is

$$\frac{1000}{g} \times 10^2, \text{ or } \frac{100000}{32},$$

that is 3125; and hence, the labouring force is the half of this, or $1562\frac{1}{2}$ dynams.

171. The principal case in which the inertia of bodies is employed to store up force is in the case of a *fly-wheel;* or *fly-weight;* in which a large mass revolving freely on an axis is connected with the machinery by which the labouring

force does its work. In this case a small force acting for a time without resistance upon the machine, may accumulate a great labouring force in the fly-weight. And when the force is thus accumulated, if it be brought to bear very suddenly and for a short time, it will overcome resistances which the small force could not overcome in any other way. Thus a man by giving gradually a rapid rotation to a machine loaded with heavy fly-balls, may punch large holes in iron plates.

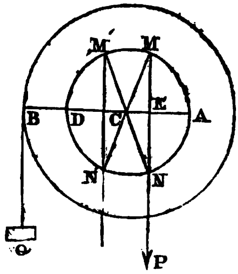
172. But a fly-wheel, though it may also be considered as a reservoir of force, is generally used for a different purpose, as an *equalizer* of force. It is principally employed when the force acts by fits and starts, or is alternately stronger and weaker, as when a foot-lathe is kept in motion by a treadle, or a fly-wheel by the piston of a steam engine. And in such cases it serves to give regularity and uniformity to the action of the machine. The force is accumulated and distributed once or twice, or it may be oftener, in each revolution of the wheel. Force is accumulated in the wheel in one part of the revolution, when the moving forces act upon it more strongly than the average; and when in turn the moving forces act more weakly, the fly-wheel goes on with the velocity acquired in virtue of its inertia, gives out more force than, for the instant, it receives, and thus brings the motion, and the labouring force, more nearly to uniformity.

The principal problem with regard to fly-wheels is to determine how the degree of uniformity in motions which they produce is regulated by their weight and dimensions, and by the forces which act upon them. This problem we now proceed to solve.

173. PROP. *An axis moved by a crank carries a given fly-wheel: to find the limits within which the velocity varies: the power acting in one direction only.*

(This is the case when a wheel is kept in motion by action on a treadle).

The force which acts at the crank is supposed to be parallel and constant. Let P be the force, MP its direction, CM the crank. Let the resistance to the motion of the machine arising from work done be equivalent to a weight Q acting perpendicularly at an arm CB. Let $CM = a$, $CB = b$.



In one revolution the labouring force of the power is (Art. 152) $P.2a$: that of the resistance is $Q.2\pi b$.

$$\text{Hence (Art. 163) } Pa = Q\pi b \ldots (1).$$

Let CM and CN be two positions in which the force P just balances the resistance Q: and let ACM be γ, CA being perpendicular to MP; therefore $P.CE = Q.CB$; or

$$Pa \cos\gamma = Qb.$$

Dividing this equation by (1), we have

$$\cos\gamma = \frac{1}{\pi}.$$

In the arc MAN, the moment of P is greater than that of Q, and therefore the velocity of rotation is increased. In the remainder of the revolution, the moment of P is not so great as that of Q, and therefore the velocity is diminished. Hence the velocity is least when the crank is in the position CM, and greatest when it is in the position CN.

Let v' and v'' be the velocities in these two positions at a distance 1 from the axis. Also let r be any radius in the fly-wheel, and Δr the element which exists at this distance: v the velocity of a point at a distance 1 from the axis.

Collect the vis viva of the system in these positions: and first, when v' is the velocity.

The velocity of any point of the fly-wheel is rv', and its mass is Δm. Hence its vis viva is $r^2v'^2\Delta m$, and the whole vis viva of the fly-wheel is $v'^2\Sigma r^2\Delta m$.

The vis viva of the resistance Q is the quantity of matter $\frac{Q}{g}$ multiplied by $b^2v'^2$, the square of the velocity.

Hence the vis viva is $\left(\Sigma r^2\Delta m + \frac{Qb^2}{g}\right)v'^2$; or if the rotatory inertia of the wheel be equal to that of a weight W at a distance b, $\Sigma r^2\Delta m = \frac{Wb^2}{g}$, and the vis viva is $(W + Q)\frac{b^2v_1^2}{g}$.

Hence the vis viva gained in passing from the position CM to the position CN is

$$(W + Q)\frac{b^2}{g}(v''^2 - v'^2).$$

But this must be, by Art. 160, equal to double the labouring force of the system expended in the same interval. Now the labouring force of P in this interval is

$$P.2ME = 2Pa\sin\gamma = 2Pa\sqrt{\left(1 - \frac{1}{\pi^2}\right)}.$$

And the labouring force of Q is

$$Q.\frac{b}{a}.\text{arc } MAN,$$

because all parts of the system revolve through the same angle: hence it is $2Qb\gamma$. And Q opposes the increase of velocity. Hence the equation is

$$(W + Q)\frac{b^2}{g}(v''^2 - v'^2) = 4Pa\sin\gamma - 4Qb\gamma \ldots (2).$$

Substitute in (2) the value of Q derived from (1) and we have

$$(W+Q)\frac{b^2}{g}(v''^2 - v'^2) = 4Pa\left\{\sin\gamma - \frac{\gamma}{\pi}\right\}.$$

The quantity $\sin\gamma - \dfrac{\gamma}{\pi}$ is a certain number; call it k, and we have

$$(W+Q)\frac{b^2}{g}(v''^2 - v'^2) = 4Pak \ldots (3).$$

Let V be the mean velocity, and let each of the extremes differ from the mean by an n^{th} part: hence

$$v' = V - \frac{V}{n}, \quad v'' = V + \frac{V}{n}.$$

$$v''^2 - v'^2 = \frac{4V^2}{n}; \text{ and substituting in (3),}$$

$$(W+Q)\frac{b^2 V^2}{gn} = Pak:$$

$$\frac{1}{n} = \frac{Pagk}{b^2 V^2 (W+Q)},$$

which is the fraction by which the velocity varies from the mean.

Cor. 1. If the fraction be given, to determine the dimensions of the fly-wheel; we have

$$\frac{Wb^2}{g} = \frac{Pakn}{V^2} - \frac{Qb^2}{g},$$

which gives the *rotatory inertia* of the fly-wheel.

Cor. 2. By (1) $4Pa - 4Q\pi b = 0$: subtracting this from (2), we find

$$(W+Q)\frac{b^2}{g}(v''^2 - v'^2)$$

$$= 4Qb(\pi - \gamma) - 4Pa\left\{1 - \sqrt{\left(1 - \frac{1}{\pi^2}\right)}\right\},$$

which expresses this: the vis viva *lost* in revolving through the arc NDM is equal to the labouring force expended in the same time.

COR. 3. If the power P, instead of being constant, vary in different positions, being still always parallel to itself, the labouring force of P in one revolution will be

$$\int P a \cos\theta\, \delta\theta,$$

which must be integrated from

$$\theta = -\frac{\pi}{2} \text{ to } \theta = \frac{\pi}{2},$$

and the result employed instead of $2Pa$.

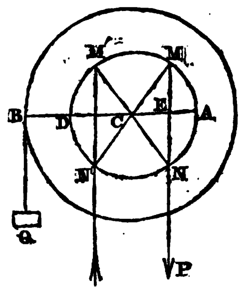
COR. 4. Since $\cos\gamma = \dfrac{1}{\pi}$, $\sec\gamma = \pi$, whence $\gamma = 71^\circ 26'$, and in parts of the radius, $\gamma = 1.2555$.

Hence $\sin\gamma = .9480$, $\dfrac{\gamma}{\pi} = .3996$, $k = .4484$.

175. PROP. *An axis moved by a crank, carries a given fly-wheel: to find the limits within which the velocity varies: the power acting alternately in opposite directions, in alternate semi-revolutions.*

(This is the case when a wheel is kept in motion by a piston which pushes up and down).

The force which acts on the crank is supposed to be parallel and constant. Let the denominations remain as before; the labouring force of the power in one revolution is $P.4a$; the labouring force of the resistance is, as before, $Q.2\pi b$. Hence



$$2Pa = Q\pi b \ldots (1).$$

Also as before, the maximum and minimum of the velocity will take place when $Pa\cos\gamma = Qb$:.

whence, dividing by (1), we find

$$\cos\gamma = \frac{2}{\pi},$$

this gives four positions of the crank, M, N, M', N', in which the velocity is alternately a minimum and a maximum. Also as before, the vis viva added to the system in passing out of the position CM into the position CN is

$$\left(\Sigma r^2 \Delta m + \frac{Qb^2}{g}\right)(v''^2 - v'^2), \text{ or } (W+Q)\frac{b^2}{g}(v''^2 - v'^2).$$

And the labouring force expended is

$$P.2ME = P.2a \sin\gamma:$$

hence equating the vis viva and the double of the labouring force, we have,

$$(W+Q)\frac{b^2}{g}(v''^2 - v'^2) = 4Pa\sin\gamma - 4Qb\gamma$$

$$= 4Pa\left(\sin\gamma - \frac{2\gamma}{\pi}\right)$$

$$= 4Pal, \text{ suppose.}$$

Hence as before,

$$\frac{1}{n} = \frac{Pagl}{b^2V^2(W+Q)}.$$

COR. If $b = a$, $2P = \pi Q$, $Q = \frac{2}{\pi}P = \frac{7}{11}P$ nearly.

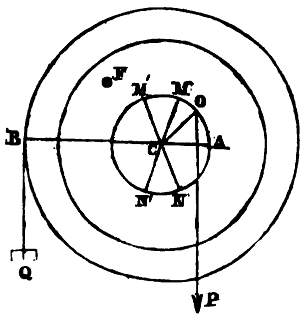
Therefore a force P acting up and down by means of a crank will produce an effect of $\frac{7}{11}P$ acting constantly at the circumference of the same circle.

176. In machines of which the parts are ponderous, every part may be considered as a fly-wheel; for every part tends to retain the motion which it has; and loses motion only in virtue of the resistances which must be overcome in order that the motion may continue. And when the moving force acts with alternate strength, each part acts upon the rest, and communicates motion to it, so as to impart or receive alternately more and less force. Hence the pressures

between the parts, and the forces which they exert on each other, will also undergo alternations.

The general case of such communication of motion may be represented by one heavy wheel, which revolves, (as we have seen that a fly-wheel does) with a velocity alternating between narrow limits, and communicates motion to another heavy wheel, by which the force is transmitted onwards towards the work to be done. If we suppose these two wheels to turn each other by means of toothed wheels fixt on their respective axes, the mutual pressure will be exerted at the point where the toothed wheels are in contact. But it will make no difference in the result, if we suppose the two heavy wheels to be upon the *same axis*; in which case we may suppose them connected by a pin. We shall determine the mutual pressure in this case; and the result applies to all cases in which one heavy wheel turns another, in a machine moved by an alternating force.

177. PROP. *An axis moved by a crank carries a given fly-wheel, and this, at a given point is pinned to another heavy wheel on the same axis: find the pressure on the pin.*



Let the force always act parallel to itself, and let OP be its direction, CA perpendicular to OP, $ACO = \theta$. Also let CO, the radius of the crank, $= a$, the force at $O = P$; CB, the radius at which the resistance acts, $= b$, and the weight which expresses the resistance $= Q$.

Let F be the force which one wheel exerts upon the other at the point F, this force being perpendicular to the radius CF; let $CF = h$. Let $\Sigma r^2 \Delta m$, the rotatory inertia of the first wheel, be equal to the rotatory inertia of a mass W_1 at a radius b, that is, $= \frac{W_1 b^2}{g}$; and in like manner let the rotatory inertia of the second wheel be $= \frac{W_2 b^2}{g}$.

Let v be the velocity at a distance 1 from the axis, $\delta\theta$ a small angle described, and δv the corresponding increment of velocity. In describing $\delta\theta$, for the first wheel,

the labouring force expended by $P = P\delta \,.\, a \sin\theta$

$= P a \cos\theta \,.\, \delta\theta$;

the labouring force consumed by $Q = Qb\delta\theta$;

the labouring force consumed at $F = Fh\delta\theta$.

But in $\delta\theta$, the vis viva generated in the first wheel is, as in Art. 174,

$$\Sigma r^2 \Delta m \left\{(v + \delta v)^2 - v^2\right\} = 2v\delta v\, \Sigma r^2 \Delta m, \text{ omitting } \delta v^2;$$

$$= \frac{W_1 b^2}{g} 2v\delta v.$$

Hence equating half the vis viva generated to the labouring force, (Art. 160)

$$\frac{W_1 b^2}{g} \,.\, v\delta v = Pa \cos\theta \,.\, \delta\theta - Fh \,.\, \delta\theta \ldots (1).$$

The vis viva generated in the second wheel, and in the resistance Q, is in like manner

$$\frac{W_2 b^2}{g} \,.\, 2v\delta v + \frac{Qb^2}{g} \,.\, 2v\delta v.$$

And equating the half of this with the labouring force consumed,

$$(W_2 + Q)\frac{b^2}{g} v\delta v = Fh \,.\, \delta\theta - Qb \,.\, \delta\theta \ldots (2).$$

Dividing (2) by (1), we have,

$$\frac{W_2+Q}{W_1}=\frac{Fh-Qb}{Pa\cos\theta-Fh}.$$

$$\text{Hence } Fh=\frac{(W_2+Q)\,Pa\cos\theta+W_1Qb}{W_1+W_2+Q}.$$

Cor. 1. Hence the pressure at F is greatest when $P\cos\theta$ is greatest, and least when $P\cos\theta$ is least.

Cor. 2. If in this case, the numerator of Fh become negative, the pressure changes its direction. When this happens, the first wheel tends to drag the second backwards, not to urge it forwards.

Cor. 3. Let the value of $P\cos\theta$, when P least urges the wheel forwards, or most urges it backwards, occur when $\theta=180^0$. Hence the pressure at F will change its direction if, at that point,

$$-(W_2+Q)\,Pa+W_1Qb \text{ be negative.}$$

Cor. 4. If P urge the wheel forwards least when $\theta=90^0$, Fh is never negative. This is the case if P act in alternate directions in alternate semicircles, as in Art. 174.

178. Prop. *The same suppositions remaining, and the force being constant, and acting in one direction only, as in Art.* 173: *to find the limits within which the pressure varies.*

As in Article 173,

$$Pa=Q\pi b,\quad\text{or}\quad Qb=\frac{1}{\pi}Pa=P\cos\gamma.$$

$$\text{Hence } Fh=Pa\,\frac{W_1\cos\gamma+(W_2+Q)\cos\theta}{W_1+W_2+Q}.$$

Cor. But since the two wheels move with the same velocity, they may be considered as one wheel, having its

inertia equal to the sum of the two. Hence, if $\frac{1}{n}$ be the fraction by which the velocity varies from the mean, we have, (Art. 173)

$$\frac{1}{n} = \frac{Pagk}{b^2 V^2 (W_1 + W_2 + Q)}.$$

Hence $$Fh = \frac{b^2 V^2}{ngk} \{W_1 \cos\gamma + (W_2 + Q)\cos\theta\}.$$

And putting for $\cos\theta$ its greatest and least values, $+1$, and -1, and for $\cos\gamma$ its value $\frac{1}{\pi}$, Fh, the moment of the pressure varies between the limits

$$\frac{b^2 V^2}{\pi ngk} \{W_1 + \pi (W_2 + Q)\},$$

and $$\frac{b^2 V^2}{\pi ngk} \{W_1 - \pi (W_2 + Q)\}.$$

Cor. 1. Let β be the value of θ when the pressure at F vanishes. Then

$$W_1 \cos\gamma + (W_2 + Q)\cos\beta = 0.$$

Whence

$$\cos\beta = -\frac{W_1 \cos\gamma}{W_2 + Q} = -\frac{W_1}{\pi (W_2 + Q)}.$$

Cor. 2. Let $W_1 = W_2 + Q$:

then $$\cos\beta = -\frac{1}{\pi} = -\cos\gamma.$$

Hence the pressure in this case vanishes at M' and N'. It is negative in the intermediate arc $M'DN'$.

Cor. 3. In this case, it appears by the proposition that the pressures at A and at D are as $1 + \pi$ and $1 - \pi$. In this case, the pressure when CO is perpendicular to CA is as 1.

CHAPTER XI.

OF SOURCES OF LABOURING FORCE.

SECTION I. GENERAL REMARKS.

179. We have seen in the preceding chapters that Labouring Force is *consumed* by work done; and may be *stored* by means of gravity, elasticity, and inertia; we have now to consider by what means it may be *generated;* or in other words, what are the *Sources* of Labouring Force.

These sources are often called *Moving Powers*. They may be enumerated as the powers of *water*, *wind*, *steam*, and other *gases*, and of *animals*, including *man*. To these we may add others, as magnetism, electricity, galvanism and chemical agencies; which however contribute very little to the general stock of force.

180. We may observe that these (excluding at present chemical and electrical agencies) resolve themselves into the effects of heat, and animal strength. For the power of water as a source, and not merely a reservoir, of labouring force, consists in its being supplied by nature at such a rate that it can be used. It flows to us from springs or from the skies, and being detained in reservoirs, is employed as labouring force. But the skies and the springs are constantly supplied by evaporation of moisture from the earth, and condensation or absorption of the moisture thus evaporated. And this evaporation is the result of heat. If clouds and springs were not constantly thus replenished the supply would fail; and water would cease to be, as it is, a

perpetual and universal source of labouring force placed by nature at our disposal.

Wind, no less than water, depends upon heat for the existence of its force. If the atmosphere were constantly of uniform temperature, there would be no currents of air, and no wind in which we could find a labouring force. The influence of heat, operating differently upon the parts of the atmosphere at different places and different times, disturbs the equilibrium of the parts, and produces currents of air from one part to another; which we can cause to operate upon machinery, and thus to do work. If the Sun were extinguished, the winds would die.

Thus the natural powers of weather and wind, no less than the steam engine, depend upon the operation of heat. The rivers continue to run, and the mill-wheel to work, in virtue of evaporation and condensation; just as the same causes keep the piston of the steam-cylinder in motion. The difference is that in the one case nature executes this constant alteration of processes, and man takes advantage of the results; in the other case man occasions and regulates the generation of the force, as well as its application. In the one case he can only store up and consume labouring force; in the other, he can put in action its source also.

181. In all the natural sources of labouring force, time must be taken into the account. The force is produced only at a certain rate; and upon an average can be consumed only at the same rate. In estimating the amount of labouring force in these cases, we must state it at so much *per second*, or *per minute*, or *per hour*, or *per day*. For example, if there be a constant stream of water 5 feet wide and 2 feet deep, flowing with a velocity of 4 miles an hour, and capable of being made to fall 10 feet, we may thus find its labouring force.

The quantity of water supplied in 1 hour is a section of ($5 \times 2 =$) 10 square feet, with a length of ($4 \times 5280 =$) 21120; therefore it is 211200 cubic feet of water: which (since a cubic foot of water is $62\frac{1}{2}$ pounds) is 13,200000 pounds per hour, or 22000 pounds per minute. And since the fall is 10 feet, the labouring force is 220000 dynams per minute.

182. In the same manner the power supplied by animals must be estimated with reference to the time. A horse can exert a labouring force of 33000 dynams per minute, or 1,980000 dynams per hour; but he cannot habitually exert this force for more than 8 or 10 hours a day. If we take 9 hours a day, the labouring force is 17,820000 per day; or very nearly 18 million dynams per day, which we may call 18 *dynamillions per day.*

Other estimates have been given by other authors. Desaguliers says, that a horse drawing a weight out of a well over a pully can raise 200 pounds for 8 hours together at the rate of $2\frac{1}{2}$ miles an hour. This gives for the labouring force 200 pounds raised 20 miles or 105600 feet: whence the labouring force of the horse is above 21 million dynams per day.

If a horse can draw a ton on a common road at the rate of $2\frac{1}{2}$ miles an hour, the friction being $\frac{1}{12}$th the load; the pressure is $\frac{1}{12}$ of 2240 pounds; and the labouring force in one minute is

$$\frac{2\frac{1}{2} \times 5280 \times 2240}{60 \times 12} = 41066 \text{ dynams.}$$

Smeaton states the labouring force of a horse to be 550 pounds raised 40 feet in one minute; this gives 22000 dynams per minute as a horse's power. But as we have said, the usual estimate is 33000 dynams per minute.

183. The power of man, as exerted in labouring force, may be estimated as in the following examples.

(1.) It is found that a man working on a tread-mill raises himself (relatively to the wheel) 10000 feet in the course of a day. He has here his weight only to raise; if we assume this to be 150 pounds, his labouring force is 1,500000, or $1\frac{1}{2}$ million dynams.

(2.) To ascend a hill 10000 feet high would be a good day's labour, this gives the same result.

(3.) In an experiment of Coulomb, porters mounting a convenient staircase 12 metres high made 66 journies in a day, carrying 68 kilograms each time. This gives 4488 kilograms raised 12 metres, together with the men's weight raised 792 metres. The man's weight being estimated at 70 kilograms, the labouring force per day is

$$4488 \times 12 + 70 \times 792 = 109296^{\mathrm{km}};$$

and as we have seen, a kilogrammetre is 7.216 dynams. Hence a man's labouring force was nearly 790000 dynams per day.

This amount is much less than the former. The carrying of weights of 68 kilograms (150 pounds) is a less advantageous mode of employing human strength than raising the weight of the body alone. The alternate ascent and descent also diminish the estimated efficiency; since the latter is not reckoned.

(4.) In pile-driving the strength of man is employed in raising a rammer which then falls by its weight. Thirty-eight labourers working 10 hours a day made in each hour 12 efforts, each effort consisting of 30 pulls, and in each pull a rammer of the weight of 587 kilograms was raised 1.45 metres.

The daily number of pulls was 3600; and hence the effort of each workman was

$$\frac{587}{38} \times 1.45 \times 3600 = 80635^{km}.3 \text{ per day};$$

which is 583678 dynams.

(5.) Desaguliers considers that a man can raise a weight of 550 pounds 10 feet high in a minute, and continue to do so for 6 hours. This gives

$$360 \times 550 \times 10 = 1{,}980000 = 2 \text{ million dynams nearly}$$

for the daily labouring force.

(6.) Smeaton thinks that a good labourer can raise 370 pounds 10 feet high in a minute; this gives 3700 for the labouring force per minute. If the labourer can do this 8 hours a day, his daily labouring force is above $1\frac{3}{4}$ million dynams.

The labouring force of men and animals is very different according to the differences of speed, and other circumstances, under which they exert it. The following rules have been given on this subject.

184. *To compare the power of animals moving with different velocities.*

The strength of men and of animals is most powerful, as pressure, when directed against a resisting object which is at rest: when the animal is in motion, the pressure which it can exert is diminished; and with a certain velocity the animal can do no work, and can only keep up the motion of its own body.

The following formula is given for the power of a man as modified by this course. Let v represent his velocity in miles per hour; then the force which he exerts in dragging forwards a load is $\frac{4}{5}(6 - v)^2$ in pounds.

Thus when v is 0, or the man is at rest, he pulls with a force of 29 pounds: when he moves at the rate of 2 miles an hour, his power of traction is reduced to 13 pounds: and if he quicken his pace to four miles an hour, he can only draw with a force of three pounds. His extreme velocity is six miles an hour.

The labouring force will be the pressure exerted, multiplied into the space described: and therefore in one hour, it is $\frac{4v}{5}(6-v)^2$.

Hence, making v respectively 0, 1, 2, 3, 4, 5, 6, we have the labouring force in these cases.

v,	0,	1,	2,	3,	4,	5,	6.
labouring force,	0,	20,	$25\frac{3}{5}$,	$21\frac{3}{5}$,	$12\frac{4}{5}$,	4,	0.

Hence the labouring force is greatest when he moves at the rate of two miles an hour.

185. The power of traction of a horse may be expressed nearly by $(12-v)^2$ pounds, where v is the number of miles per hour which the horse is moving. If the rate be 4 miles an hour, the power of traction would be 64 pounds. Also the labouring force would be this pressure, multiplied into the space described. At 4 miles an hour, the space described in one minute is $\frac{5280}{15}$, or 352 feet: and therefore the labouring force in a minute is 22528 dynams. If the horse were to move with a velocity of one mile an hour, the labouring force would be, in like manner, $121 \times 88 = 10648$ dynams per minute; and with a velocity of ten miles an hour, the labouring force would be only 4×880, or 3520 dynams per minute.

A wagon on a turnpike road, loaded to the amount of 8 tons, may be drawn by 8 horses at the rate of $2\frac{1}{2}$ miles an

hour, the horses working for 8 hours daily. Thus the labouring force of a horse in this way will amount to 1 ton transported twenty miles a day; or if the friction be one-twelfth of the load, this amounts to a pressure of $1\frac{2}{3}$ ton through a space of 1 mile, or in pounds and feet, to 19,712000 dynams per day, and 41066 dynams for 1 minute. In this case, the labouring force of a horse is nearly 20 million dynams per day.

A mail coach weighing 2 tons and travelling at the rate of 10 miles an hour, may be worked on a line of road in both directions by a number of horses equal to the number of miles. The labouring force of each horse would amount to 2 tons drawn 2 miles daily, or 1 ton drawn 4 miles. Thus with this great velocity, the work done is only one-fifth of what it was in the other case. The horse-power in this case is 8215 dynams per minute, or four million dynams per day.

186. There is one mode in which the strength of man and animals is employed, in which force exerted by them is not directly comparable with labouring force in other cases: namely, when they *carry* burthens along a horizontal road. A man can draw a certain burthen on a cart or sledge on a given road at a certain rate of velocity, and the labouring force thus exerted may be estimated when the resistance of the road to the carriage is known. The load he can carry with the same speed will be much less: and the difference will not be properly estimated by supposing the resistance to horizontal motion to be greater in this case than in the other. The most proper way of considering the case is this. A certain portion of the man's strength is employed in merely supporting the load; for even if a man stands still, he must exert his strength in order to support a heavy load; and the load may be so great that his strength

shall be barely sufficient, or not sufficient, to support the load. But the strength so exerted is not labouring force, measured by the load and the space moved through. Yet supposing the man able to support the load, the difficulty of carrying it through a given horizontal space increases with the increase of the load. Hence we must suppose that a certain portion of the man's force is employed in supporting the load; and that he has only the remainder to employ as labouring force, measured by the difficulty of carrying the load and the space through which it is carried. Therefore the power of a man to carry a load horizontally will depend upon his strength in a more complex manner than in the cases where the measure of labouring force is applicable immediately. The same may be said of the power of horses, or other animals, to carry loads.

SECTION II. POWER OF STEAM.

187. STEAM acts by means of its elasticity, and its capability of being generated and condensed by heat and cold. Its labouring force may be measured in the same manner as that of any other elastic body, but it has peculiar properties which make a more special consideration necessary.

Steam is used as a mechanical agent in various ways, of which the principal will be exemplified by considering, (1.) Atmospheric Engines, in which the pressure of the atmosphere is made available by the condensation of steam: (2.) Condensing Engines, in which the motion of the piston each way is produced by steam on one side of it, rendered available by the condensation on the other side: in this case the steam may have a higher elasticity than the atmospheric air; (3.) High-pressure Engines, in which the steam must have a greater elasticity than the atmosphere, since it is made

to urge the piston against the pressure of the atmosphere without condensation.

(1.) *Atmospheric Engines.*

188. In Atmospheric Engines, a piston, moveable in a vertical cylinder, rises to the top of the cylinder when steam is admitted below it, of the same elasticity as the atmosphere; the piston being slightly overbalanced. The steam is then condensed, and the piston is urged, by the whole pressure of the atmosphere, to the bottom of the cylinder. The steam being again admitted under the piston, the piston rises to the top of the cylinder; and so on.

In this case, the efficiency of the machine is that corresponding to the descending stroke: the raising of the piston does not produce any power, but is necessary to the continuance of the alternating motion.

189. *To find the theoretical duty of an atmospheric Steam Engine corresponding to one cubic foot of water.*

The space occupied by steam, of the temperature of boiling water, and consequently of the elasticity of the atmosphere, is 1711 times the bulk of the water which produces it. (Tredgold on the Steam Engine, Art. 302.) Hence one cubic foot of water produces 1711 cubic feet of steam of the pressure of the atmosphere. This pressure is, at its mean, 2120 pounds on a square foot: hence the whole pressure is

$$1711 \times 2120 = 3{,}627320 \text{ lbs.}$$

supposing the steam to occupy a space a foot high and 1711 feet in horizontal surface. And when this is condensed, the whole of this pressure acts through 1 foot; and therefore the labouring force is 3,627320 dynams.

If the steam occupy a prismatic space h feet high, the horizontal surface will be $\dfrac{1711}{h}$, and the pressure on the

piston $\frac{1711}{h} \times 2120$; and when the steam is condensed, the piston will move through h feet; whence 1711×2120 dynams will still be the labouring force of the machine.

If the piston be overbalanced by an excess of pressure p in its ascent, and if q be the pressure of the atmosphere on the piston, h the length of the stroke; there is a labouring force ph capable of doing work in the ascent of the piston; and the labouring force exerted in the descent is $(q-p)h$: hence the whole labouring force of the double stroke is qh; which is the same as when there is no available excess of balancing force.

If p be $\frac{1}{2}q$, the labouring force in the ascent and descent will be equal, namely $\frac{1}{2}qh$ in each case. This is a mode in which atmospheric engines are often employed.

190. PROP. *To find the labouring force of an atmospheric Steam Engine, corresponding to* 1lb. *of coal.*

It appears (Tredgold, Art. 191,) that it requires from 7 to 10 lbs. of Newcastle coal to form a cubic foot of water into steam of the elasticity of the atmosphere. Mr. Watt states, that a bushel of coal (84 lbs.) would evaporate 10 cubic feet of water (Wood on Rail Roads, p. 353). Mr. Davies Gilbert gives 14 cubic feet as the water evaporated by one bushel.

Taking Mr. Watt's statement we have 8.4lbs for the coal requisite to evaporate 1 cubic foot.

Hence dividing the labouring force of 1 cubic foot of water (Art. 189.) by 8.4, we have 431824 for the labouring force of 1lb. of coal.

Or, since a bushel of coal reduces 10 cubic feet of water to steam, we have, by the last Article, 36,273200 for the labouring force of a bushel of coal in an atmospheric en-

gine; and this, divided by 84, gives 431824 for the labouring force of 1lb.

Mr. Gilbert gives 39,361000 for the labouring force of a bushel of coal employed in an atmospheric engine.

From this must be deducted the efficiency requisite to work the air pump, which is usually about one eighth of the whole; and the resistance arising from imperfection of the vacuum, as well as the friction.

(2.) *Condensing Engines.*

191. In these the motion of the piston to and fro is produced by the alternate action of steam upon each side of the piston, the steam on the other side being condensed. The steam here may be, and often is, of a greater elasticity than the atmosphere.

We shall not attempt here to find the labouring force of a given quantity of coal. It appears by the results that this is a very advantageous way of using steam.

The labouring force of such engines has been much increased by two expedients:—First, by raising the temperature of the steam above that of boiling water, by which its elasticity is made greater than that of the atmosphere; this proceeding being of course accompanied by an increased expenditure of fuel.

Secondly, by causing the steam to act expansively; that is, by stopping the influx of steam when the piston has moved along part of the length of the cylinder, so that the remainder of the stroke is produced by the expansive power of the steam already admitted.

By these and other improvements in the machinery and economy of condensing steam engines, the labouring force of a bushel of coal has been carried much beyond the limit

calculated above for atmospheric engines. In 1829 an engine in Cornwall, with a cylinder of 80 inches diameter, had a *duty*, that is, an actual labouring force, of 75,628000, and several others approached this amount.

We may thus consider 70 million dynams as below the actual labouring force of a bushel of coal, and 840000 dynams as the actual labouring force of a pound of coal, working to great advantage in a condensing engine.

Hence one pound of coal is capable of raising itself through 840000 feet, or about 160 miles.

(3.) *High-pressure Engines.*

192. If a piston be moveable in a cylinder as before, and if steam of a greater elasticity than the atmosphere be admitted on one side of the piston, the cylinder on the other side having an opening into the atmosphere, the piston will be moved to the open end. If then this end be closed and steam of the same kind admitted into it, the other end being opened, the piston will move back again; and by a continuation of this process an alternating motion may be produced.

Prop. *To find the labouring force of a High-pressure Engine.*

In order to determine the labouring force of a High-pressure engine corresponding to a given quantity of coal, it is necessary to know the quantity of steam, of the elasticity employed in the engine, which the given fuel would produce. Moreover it appears that in the transmission of high-pressure steam from the boiler to the cylinder a portion of its elasticity is lost in its passage through the valves.

If we know the elasticity of the steam in the cylinder, we may determine the labouring force, as in the following example.

In a locomotive engine the surface of the pistons was 127.2 square inches; the elasticity of the steam in the boiler 50lbs. per square inch more than the pressure of the atmosphere, the length of the stroke 2 feet, the diameter of the travelling wheels 37 inches: to find the labouring force expended in travelling 388 yards. (Wood on Railroads, p. 346.)

The pressure on the pistons is 6360lbs., and the space described by the piston in a double stroke is 4 feet; and hence the labouring force for one such stroke is 6360×4 dynams.

The diameter of the wheels being 37 inches, the circumference is 116.24. And the length of path described is $3 \times 388 \times 12$ inches; hence the number of revolutions of the wheel is $3 \times 388 \times 12 \div 116.24 = 120$ revolutions. And to each revolution corresponds a double stroke of the piston. Therefore the whole labouring force expended is

$$6360 \times 4 \times 120 = 3{,}052800 \text{ dynams,}$$

supposing the steam in the cylinder to be of the same elasticity as that in the boiler.

It appeared in the experiment to which this calculation refers that the total amount of the resistances was 1829 lbs., which was moved over 3×388 feet: therefore the work done was $1829 \times 3 \times 388 = 2{,}128956$ dynams.

The excess of the labouring force expended over the work done is to be attributed partly to resistances which have been left out of the account, and partly to the different elasticity of the steam in the boiler and in the cylinder.

In engines of this kind it was found that it required from 18 to 21 lbs. of coal to evaporate a cubic foot of water.

More recently it has been found (M. de Pambour on Locomotive Engines, p. 298, second edition) that from 7 to $12\frac{1}{2}$ lbs. of coke are consumed in vapourising 1 cubic foot of water in locomotive engines. The mean of twenty-two experiments gives 11.7 lbs.

193. The following may be taken as another example of this calculation. In a small high-pressure engine the cylinder was 8 inches in diameter, with a stroke of $4\frac{1}{2}$ feet: it worked a pump $18\frac{1}{2}$ inches diameter, and $4\frac{1}{2}$ feet stroke, which raised water 28 feet high. The engine consumed 80 lbs. of coal per hour, working 18 strokes per minute.

The section of the column of water $= 18.5 \times 18.5 \times .7854 = 268.8$ square inches; and the pressure of a column of water 1 foot high on a square inch is .434 lbs. Hence the weight of the column of water is

$$268.8 \times .434 \times 28 = 3266.5 \text{ lbs.}$$

The motion of the piston per minute is $4\frac{1}{2} \times 18 = 81$ feet, or 4860 feet per hour: and this multiplied by 3266.5 gives 15,875190 dynams for the hourly labouring force.

And dividing by 80, we have for the labouring force of 1 lb. of coal, $198439\frac{1}{2}$ dynams.

In this case, since the stroke of the pump and of the piston are equal, the pressures must be equal, in order that they may balance each other. Hence the pressure on the piston is 3266.5 lbs; and since the area is $8 \times 8 \times .7854 = 50\frac{1}{4}$ square inches; the effective pressure per inch is 65 lbs. This is the excess of the elasticity of the steam above that of the atmosphere.

194. In locomotive engines moving uniformly, the pressure is employed in maintaining the velocity; and since the resistance is independent of the velocity, the pressure on the piston is also independent of the velocity. Hence, when

the same engine moves with different velocities, the elasticity of the steam in the cylinder is nearly the same: the principal difference is the rapidity of its generation.

195. *To compare the efficiency of a given quantity of coal in a Locomotive and in an Atmospheric engine.*

It appears that the best modern locomotive engines on railroads require half a pound of coal per ton per mile.

If we suppose the friction to be $\frac{1}{240}$ of the weight, the friction for a ton is $9\frac{1}{4}$lbs.: and in one mile the work done is $9\frac{1}{4} \times 5280 = 48840$ dynams for half a pound of coal, or 97680 for one pound of coal.

By M. de Pambour's experiments, (Locom. Eng. p. 312, sec. ed.) it appears that to draw a ton one mile on a level, the coke consumed varied from .34 to .66lbs: the mean being about $\frac{1}{2}$lb.

Also it appeared (p. 219) that several engines, (*The Sun*, *Firefly*, *Vulcan*, *Fury*, *Leeds*, and *Jupiter*) of the average weight of 8 tons, had a mean friction of 104lbs. Taking this as the basis, it appears that the friction is 13lbs. per ton. And hence the work done in drawing 1 ton a mile is 5280×13 dynams. And the labouring force corresponding to 1lb. of coke is therefore by these experiments $5280 \times 26 = 137280$ dynams. But the engine *Star* of the weight of 11 tons, had a friction of 176lbs. or 16lbs. per ton. Hence for this engine the labouring force per pound of coke was $5280 \times 32 = 168960$ dynams.

The labouring force for one pound of coal in an atmospheric engine is, as we have seen, (Art. 190,) 431824 dynams, which is from $2\frac{1}{2}$ to $4\frac{1}{2}$ times as much as the other.

196. It appears that the fuel consumed is very nearly proportional to the work done in locomotive engines, at

whatever rate they travel. The speed depends upon the degree of rapidity with which the water can be converted into steam of the given elasticity. Hence the speed is increased by increasing the surface of the boiler which is exposed to the fire; and by increasing the draft of the furnace.

One mode of producing the latter effect, which has been found very efficacious in practice, is the throwing the waste steam up the chimney when it is driven out of the cylinder. There appears to be theoretically no limit to the velocity which may thus be attained.

197. Practically speaking, the velocity of a locomotive engine is, as we have said, limited by the rate at which the water can be converted into steam. Hence we shall add the following Proposition.

PROP. *Given the ratio of evaporation, and the pressure of the steam in the boiler: to find the velocity of the engine, moving uniformly.*

Let the whole resistance to the motion of the carriage, arising from the friction of the road and of the engine, be R; let D be the diameter of the driving wheels, C the diameter of each of the two steam cylinders, l the length of the stroke, m the pressure of the atmosphere on a square inch.

Let q be the pressure of the steam in the cylinder on a square inch: then, since the area of the two pistons is $\frac{1}{2}\pi C^2$, $(q - m)\frac{1}{2}\pi C^2$ is the effective pressure on the piston. And by Art. 46, the principle of vertical velocities may be here applied; hence

$$R \times \pi D = (q - m)\,\tfrac{1}{2}\pi C^2 \times 2l,$$

$$q = \frac{DR}{C^2 l} + m.$$

Let p be the pressure of the steam per inch in the boiler:

Let s represent the rate of evaporation, that is, the number of cubic feet the boiler is able to evaporate in a minute at the pressure p:

And let n be the ratio of the volume of steam at the pressure p, to the volume of water.

Hence ns will be the volume of the steam generated in the boiler in a minute.

Then since q is the pressure of the steam in the cylinder, the steam, in going from the boiler to the cylinder, passes from the pressure p to the pressure q, and (by known principles) changes its volume in the inverse ratio of the pressures. Hence, in the cylinder, the volume of the steam will be $\frac{nsp}{q}$.

This volume of steam passes through the cylinders in a minute. Hence if we divide it by the section of the cylinders, or $\frac{1}{2}\pi C^2$, we shall have its mean velocity. Hence the mean velocity of the steam, and therefore of the piston, is

$$\frac{2nsp}{\pi C^2 q}.$$

Now the velocity of the carriage is to that of the piston as πD to $2l$; therefore the velocity of the carriage is

$$\frac{nspD}{C^2 ql};$$

or putting for q its value, the velocity is

$$\frac{nspD}{DR + mC^2 l}.$$

198. Ex. In an engine of which the load was 100 tons, the cylinders were 11 inches diameter, the stroke 16 inches, the wheels 5 feet; the effective pressure of the steam in the

boiler 50lbs. per square inch, the effective evaporating power 42 cubic feet of water per hour, or 0.7 cubic feet per minute. Find the rate of travelling.

This water is immediately transformed in the boiler into steam at the effective pressure of 50lbs. per square inch, or (since the pressure of the atmosphere is 15 lbs.) at the total pressure of 65lbs. per square inch.

According to tables founded on experiment, steam generated under a total pressure of 65 lbs. per square inch occupies 435 times the space of the water which produced it. Thus the water expanded in each minute formed a volume of steam of $0.7 \times 435 = 304$ cubic feet.

It appeared by calculations respecting the engine in question, that with a load of 100 tons, the resistance was 46lbs. per square inch of the piston. Hence the steam, in passing from the boiler to the cylinders, was reduced from the pressure 65 to 46. The volume being increased in the same ratio becomes

$$304 \times \frac{65}{46} = 430 \text{ cubic feet,}$$

which passes through the cylinders every minute.

Now the area of the section of the two cylinders is 190 square inches, or 1.32 square feet. Hence the above steam passes through the cylinders with a velocity of

$$\frac{430}{1.32} = 326 \text{ feet per minute,}$$

which is the mean velocity of the piston.

The velocity of the carriage is to that of the piston in the proportion of the circumference of the wheel to the double stroke; that is, as 5.887 to 1. Hence the velocity of the carriage is

$$5.887 \times 326 \text{ feet per minute; or}$$

$$5.887 \times 326 \times \frac{60}{5280} = 21.83 \text{ miles per hour.}$$

SECTION III. REGULATION OF POWER.

199. If the supply of power be variable either absolutely or with regard to the work to be done, this variation will cause variations in the velocities and pressures which take place in the machinery. We have seen in the last Chapter, that after the power has assumed the form of vis viva, it may be stored up as in a reservoir, and thus the rate of working may be equalized. But the rate of working may also be equalized by regulating the supply of power according to the work to be done. Thus in a water-mill, by opening more or less the flood-gate of the dam we increase or diminish the rate of supply of the water, and therefore of the power. In like manner in a steam engine, the supply of power is greater or less as we open more or less the steam-valve: or again, as we cause the fire to burn more or less briskly; and by applying such means we may *regulate* the action of the machines here spoken of.

200. But instead of increasing or diminishing the supply of power by extraneous means, we may apply to the machine such contrivances that it shall regulate its own action. We may construct mechanism such that when the machine goes slower than we wish it to go, it shall itself increase the supply of power, and vice versâ. Thus a mill-wheel, or the wheel of a steam engine, may have mechanism connected with it, such that when the velocity diminishes beyond a certain rate, the flood-gate of the mill, or the steam-valve of the engine, shall be opened wider: or again, that some aperture shall be opened, by which the draft of the fire shall be increased.

A piece of mechanism used for this purpose in the steam engine is called the *governor*.

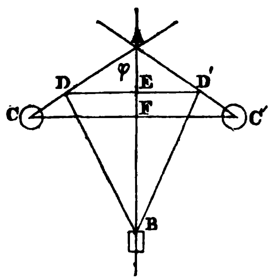
201. The governor which we shall treat of consists of balls at the end of two rods, moving by hinges in vertical

planes on opposite sides of a vertical stem, which revolves with the motion of the machine. These balls revolve horizontally by the revolution of the vertical stem. The rods are connected by other rods and hinges with a slider, which by sliding upwards diminishes, and by sliding downwards increases the supply of steam.

It is plain that such a governor will regulate the rate of going of the engine. For if the rate tend to increase, the balls will be thrown further from the vertical axis by the increase of centrifugal force; the slider will be raised; the supply of steam will be diminished; and the increase of velocity will be less than it would be without the effect of the governor.

202. Each rod which carries one of the balls, describes a conical surface in its motion, and is called a *conical pendulum*. The two are introduced, rather than one, only to destroy lateral pressures.

PROP. *In a governor, consisting of a conical pendulum, to find the position of the slider, knowing the angular velocity.*



Let A be the upper hinge, AB the vertical stem, C, C', the revolving balls, BD', BD the connecting rods, B the slider.

Let $AD = a$, $BD = b$, $AC = c$, $AB = y$, $CAB = \phi$, angular velocity $= \omega$. Then the centrifugal force of C is $CF \times \omega^2$ (Mech. 97).

Now the forces by which C is acted upon are the centrifugal force in the horizontal direction FC, and the vertical force of gravity g. And the resultant of these two forces must be in the direction AC; for otherwise the resultant would tend to turn AC round A. Hence (neglecting the inertia of the rods),

$$\text{centrifugal force } (= CF \cdot \omega^2) : g :: CF : AF;$$

$$\text{whence } g = AF \times \omega^2 = c \cos \phi \cdot \omega^2.$$

$$\text{Hence } \omega^2 = \frac{g}{c \cos \phi}; \quad AE = \frac{ag}{c\omega^2}.$$

$$\text{And} \quad AE + EB = AE + \sqrt{\{BD^2 - DE^2\}}$$

$$= AE + \sqrt{\{BD^2 - DA^2 + AE^2\}},$$

$$\text{or } y = \frac{ag}{c\omega^2} + \sqrt{\left\{b^2 - a^2 + \frac{a^2 g^2}{c^2 \omega^2}\right\}}.$$

Hence y is found from ω.

Cor. 1. To find ω from y, we have

$$BE^2 = BD^2 - DE^2,$$

$$\text{or} \quad (y - a \cos \phi)^2 = b^2 - a^2 \sin^2 \phi,$$

$$\text{whence } y^2 - b^2 + a^2 = 2ay \cos \phi,$$

$$\omega^2 = \frac{g}{c \cos \phi} = \frac{2agy}{c\{y^2 - b^2 + a^2\}}.$$

Cor. 2. If $b = a$, the expressions become

$$y = \frac{2ag}{c\omega^2}; \quad \omega^2 = \frac{2ag}{cy}.$$

Cor. 3. In this case, suppose ω to receive a small given increment $\delta\omega$, and y a small increment δy,

$$\delta y = -\frac{4ag}{c\omega^3} \delta\omega.$$

Hence y is diminished by a quantity which is inversely as the cube of the angular velocity.

COR. 4. Since $\cos\phi = \frac{g}{c\omega^2}$, $\sin\phi \, . \, \delta\phi = \frac{2g}{c\omega^3}\delta\omega$.

$$\delta\phi = \frac{2g}{c\omega^3 \sin\phi}\delta\omega = \frac{2\cos\phi}{\sin\phi}\frac{\delta\omega}{\omega}.$$

Ex. 1. A governor is revolving once a second; the length of the rods being 2 feet; find the angle which they make with the vertical.

$$\cos\phi = \frac{g}{c\omega^2}, \text{ and in this case } \omega = 2\pi;$$

$$\text{hence } \cos\phi = \frac{g}{8\pi^2} = \frac{4}{10}, \text{ nearly;}$$

$$\text{whence } \phi = 66^\circ, \text{ and } \sin\phi = \frac{9}{10}, \text{ nearly.}$$

Ex. 2. In this case the angular velocity is increased by $\frac{1}{10}$ of its own amount: to find the change in the angle.

Here, $\frac{\delta\omega}{\omega} = \frac{1}{10}$; whence $\delta\phi = \frac{2\cos\phi}{\sin\phi}\frac{\delta\omega}{\omega} = \frac{8}{9}.\frac{1}{10}$.

And this in degrees is $\frac{8}{90} \times \frac{180}{3.14} = 5^\circ$ nearly.

CHAPTER XII.

OF IMPACT.

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203. WHEN one body strikes or impinges upon another the velocity of one or both is *suddenly* altered. In this case the general principles of mechanical action undergo modifications which we shall here explain.

If the bodies do not separate after the impact, they must be inelastic: for as we have seen (Mech. 136, 137,) perfectly elastic bodies separate after the impact with the same relative velocity with which they approached each other before the impact, and imperfectly elastic with a smaller. Thus we have impact of inelastic bodies, for instance, when a hammer drives a nail, or a ram drives a pile.

204. In cases of impact the effect is produced by a sudden and brief pressure of great intensity. The length of time occupied by the pressure, and the law according to which its variation goes on during this time, will depend upon the material of the body, and other circumstances. When one body impinges upon another, so that the two go on together after the impact, we may conceive the pressure which takes place between the two to vary steadily, till it attains its ultimate value, and the bodies go on without further mutual action; or we may suppose that the action is of the nature of a series of vibrations constantly diminishing, and at last vanishing.

205. In all cases in which the bodies do not separate after the impact, we suppose some permanent change to take

place in each body, in consequence of the blow. The body is *disfigured*, externally or internally; although this disfigurement may be so small as not to be perceptible to the senses. The smallness of the disfigurement arises from the great intensity of the forces which resist it (the hardness of the body). But in most cases, after many impacts, the disfigurement is perceptible.

206. *When a body* A *strikes a body* B, *the mutual pressure must, at every instant, be equal upon one and the other.* If we suppose the law of compression and the measures which it involves to be known, we may obtain a relation between the compression of the one body and the other.

207. Prop. *Let the linear compression in the direction of the pressure be as the pressure: to find the proportion of the compression of each body.*

Let h be the length of the body A in the direction of the pressure; p the portion by which h is diminished by the pressure; a the section of A perpendicular to h; E the modulus of elasticity of the material with regard to impact*. Let the quantities h, p, a, E, be for B, k, q, b, E'.

The force of each fibre of A parallel to h is the weight of a column $\frac{Ep}{h}$; hence the whole force (supposing all these fibres equally compressed,) is the weight of a column $\frac{Eap}{h}$. In like manner the force exerted by the fibres of B is the weight of a column $\frac{E'bq}{k}$. Hence, since these pressures are equal,

$$\frac{Eap}{h} = \frac{E'bq}{k}; \quad \frac{p}{q} = \frac{E'bh}{Eak}.$$

* We retain the term *Modulus of Elasticity* in these cases, though we call the bodies *inelastic*.

For $\frac{Eap}{h}$ we may put αp, and for $\frac{E'bq}{k}$ we may put βq. This being done, α and β may be called the ***hardness*** of the bodies A and B; for they are inversely as the space through which a given force would compress the surfaces. But α, β, depend upon the dimensions as well as the material, as we have seen.

208. PROP. *When one body impinges on another, which is immoveable, the force exerted is greater in proportion as the bodies are harder.*

Let, as before, p, q, be the compression of A, B, respectively; and αp, βq, the forces exerted by the striking body, and the body struck. Therefore, as we have seen, $\alpha p = \beta q$.

Let A be the weight of the striking body or *hammer*, V its velocity at the instant of first contact, s the space described by its center since that time. This space (B being immoveable) is described in consequence of the compression of the two bodies, therefore $s = p + q$.

$$\text{But } \alpha p = \beta q; \; q = \frac{\alpha p}{\beta}, \; s = \frac{\alpha + \beta}{\beta} p;$$

$$p = \frac{\beta s}{\alpha + \beta}, \; q = \frac{\alpha s}{\alpha + \beta}.$$

The force on A is βq, and the quantity of matter in A is $\frac{A}{g}$; hence the force which retards A is

$$\frac{g\beta q}{A} = \frac{\alpha\beta}{\alpha + \beta}\frac{gs}{A}.$$

Hence by the equation, $v\frac{dv}{ds} = f$, (v being the velocity of A,)

$$v\frac{dv}{ds} = -\frac{\alpha\beta}{\alpha+\beta}\frac{gs}{A}; \quad v^2 = C - \frac{\alpha\beta}{\alpha+\beta}\frac{gs^2}{A}:$$

and since $v = V$, when $s = 0$, $v^2 = V^2 - \dfrac{\alpha\beta}{\alpha+\beta}\dfrac{gs^2}{A}$.

$$\text{Hence, when } v = 0,\ s = V\sqrt{\frac{(\alpha+\beta)A}{\alpha\beta g}};$$

And the pressure at this moment

$$= \beta q = \frac{\alpha\beta s}{\alpha+\beta} = V\sqrt{\frac{\alpha\beta A}{(\alpha+\beta)g}}.$$

At this moment the whole motion of the hammer is destroyed; and the compression is greatest, and the force greatest. The force increases from the first contact, when it is 0, till it is

$$V\sqrt{\frac{\alpha\beta A}{(\alpha+\beta)g}} = V\sqrt{\frac{A}{\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)g}}.$$

Now this is greater as $\dfrac{1}{\alpha}$ is less, or α greater; and also as $\dfrac{1}{\beta}$ is less, or β greater. Therefore the force exerted is greater as the hardness of either body is greater.

Cor. 1. If $\beta = \alpha$, the greatest compression is

$$= V\sqrt{\frac{2A}{\alpha g}};$$

$$\text{the greatest pressure is} = V\sqrt{\frac{A\alpha}{2g}}.$$

Cor. 2. In this case put for α its value $\dfrac{E\alpha}{h}$;

also put $A = ah$, E being expressed by a column of the material;

Therefore the greatest compression

$$= V\sqrt{\frac{2h^2}{Eg}} = Vh\sqrt{\frac{2}{Eg}},$$

$$\text{the greatest pressure} = Va\sqrt{\frac{E}{g}}.$$

COR. 3. Let the velocity be that acquired down a height H by gravity: here $V = \sqrt{2gH}$.

$$\text{Hence the greatest compression} = h\sqrt{\frac{4H}{E}};$$

$$\text{the greatest pressure} = a\sqrt{2EH}.$$

Ex. 1. Let a hammer of iron be $\frac{1}{4}$ foot high, and fall from a height of 8 feet on an iron anvil of the same hardness; to find the compression and pressure. ($E = 9,000000$).

$$\text{The greatest compression} = \frac{1}{4}\sqrt{\frac{32}{9,000000}} = \frac{\sqrt{2}}{3000}$$

$$= \frac{1}{2143} \text{ of a foot.}$$

$$\text{The greatest pressure} = a\sqrt{(2 \times 8 \times 9,000000)}$$

$$= a \times 12000,$$

that is, it is the weight of a column of iron 12000 feet high, and of the same section as the hammer.

Ex. 2. Let the modulus of elasticity of the anvil be the same as that of the hammer.

Let a, b, be the sections of the hammer and anvil, h, k, their heights; then (Art. 207)

$$\alpha = \frac{Ea}{h}, \quad \beta = \frac{Eb}{k}: \quad \frac{\beta}{\alpha} = \frac{bh}{ak} = m, \text{ suppose.}$$

Hence the greatest compression is

$$V\sqrt{\left(1 + \frac{1}{m}\right)\frac{A}{g}}.$$

Thus if the anvil be 8 times as high as the hammer, and its section 16 times as great as that of the hammer, $\frac{\beta}{\alpha} = 2$: the anvil is, as to linear compression, twice as hard as the hammer.

209. In the above case the body struck was supposed to be kept at rest by an immoveable obstacle. In this case the pressure is the greatest possible. If the body struck can move, the force will not be so great. We will consider the effect of impact in moving a body in opposition to a uniform force. And first, let the body struck be so small that its weight and inertia may be neglected, as when a hammer drives a nail, the friction being supposed uniform.

Prop. *To find how far a given hammer will drive a nail.*

Taking the same letters as before, and F being the friction, it is evident that the nail will not stir till the compression of the nail and hammer produce a pressure upon the nail $= F$. This will occur when

$$\alpha p = \beta q = F; \text{ that is } \frac{\alpha\beta s}{\alpha + \beta} = F,\ s = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) F.$$

Hence, by the last Proposition, if v be the velocity of the hammer when the nail begins to move,

$$v^2 = V^2 - \frac{gs^2}{A\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)} = V^2 - \frac{g}{A}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) F^2.$$

When the nail has begun to move, the hammer and nail move on resisted by the uniform force F, till the momentum of the hammer is destroyed. Let s' be the space through which they move. Then, since the uniform force which retards A's motion is $\frac{Fs}{A}$, we have

$$s' = \frac{v^2}{\frac{2Fg}{A}} = \frac{A}{2g}\left\{\frac{V^2}{F} - \frac{g}{A}\left(\frac{1}{a} + \frac{1}{\beta}\right) F\right\}.$$

Cor. 1. This space will not have any positive value, and the blow will not move the nail at all,

$$\text{if } V^2 \text{ be less than } \frac{g}{A}\left(\frac{1}{a} + \frac{1}{\beta}\right) F^2.$$

Cor. 2. If V be very great, the second part of the value of s' will be small compared with the first, and the space through which the nail moves will be nearly independent of the hardness a, β. But if V be barely sufficient to move the nail, a small increase in the hardness of the hammer or nail will much increase the space through which the nail moves.

Cor. 3. If, while friction prevents the nail from moving *in* the block into which it is driven, it move *with* the block on account of the yielding of the block, this will produce the same effect as the softness of the nail; that is β, the hardness of the nail will be diminished. Hence a nail will be driven farther, cæteris paribus, into a substance that does not yield than into one that does.

Cor. 4. Since at the limit of motion

$$V^2A = gF^2\left(\frac{1}{a} + \frac{1}{\beta}\right),$$

when the bodies are very hard, or a, β, very great, a small hammer and small velocity will produce the effect of a very great pressure F. Thus it is found that a sledge-hammer driving hard oak pegs, produces as great an effect as a pressure of 70 tons.

Cor. 5. If V be the velocity acquired down H,

$$s' = \frac{HA}{F} - \left(\frac{1}{a} + \frac{1}{\beta}\right)\frac{F}{2}.$$

COR. 6. If $\beta = a$, the hardness of the nail as measured by linear yielding, is equal to that of the hammer.

$$\text{and } s' = \frac{A}{2g}\left(\frac{V^2}{F} - \frac{gF}{2Aa}\right) = \frac{HA}{F} - \frac{F}{4a}:$$

$$\text{or since } a = \frac{Ea}{h},\ s' = \frac{HA}{F} - \frac{Fh}{4Ea}.$$

Here A, F, and $4Ea$ are all weights of known masses.

COR. 7. If the hardness of the hammer and the nail were perfect, a and β would be infinite, and

$$s' = \frac{HA}{F}.$$

Hence the latter term $\frac{F}{4a}$, is the defect of the space through which the nail is driven, due to the imperfect hardness of the material; that is, this defect is

$$\frac{Fh}{4Ea};$$

or if a and β be different, E being the same for both, the defect is

$$\frac{F}{2}\left(\frac{1}{a} + \frac{1}{\beta}\right) = \frac{F}{2E}\left(\frac{h}{a} + \frac{k}{b}\right).$$

Ex. An iron hammer drives a nail *of equal hardness* in a block, such that the resistance to the nail is equal to n times the weight of the hammer: and the elasticity of the hammer is exprest by a column of iron 10 million times the height of the hammer; to find how far the nail is driven. Here $F = nA$, $Ea = 10{,}000000A$.

$$\text{Hence } s' = \frac{H}{n} - \frac{h}{40{,}000000},$$

of which the second term is the defect arising from imperfect hardness.

210. In the next instance we take into account the weight and inertia of the body moved.

PROP. *A pile of weight,* B, *is driven by a ram or hammer* A, *impinging with a velocity* V: *the friction being* F: *to find the motion.*

Let s' be the space through which the aggregate compression takes place before the pile moves. Then, as in last article, this will occur when the force downwards equals the resistance, or $\beta q + B = F$: Hence, as in Art. 207,

$$\frac{\alpha\beta s'}{\alpha + \beta} = \beta q = F - B, \quad s' = \frac{\alpha + \beta}{\alpha\beta}(F - B).$$

And as before, v being the velocity of the ram,

$$v\frac{dv}{ds} = -\frac{g\alpha\beta s}{(\alpha + \beta)A} + g,$$

$$v^2 = V^2 - \frac{g\alpha\beta s^2}{(\alpha + \beta)A} + 2gs, \text{ because } v = V \text{ when } s = 0.$$

If V' be the velocity of the ram when the pile begins to move; putting s', or its value, for s,

$$V'^2 = V^2 - \frac{g}{A}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)(F - B)^2$$

$$+ 2g\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)(F - B) \ldots (1.)$$

Now as the inertia of the pile resists the communication of motion, the compression will go on increasing after this time. Let x be the space through which the ram has moved since the first contact, y the space through which the pile has moved; then $y = x - s$, and $s = x - y$.

For the motion of the ram,

$$\frac{d^2x}{dt^2} = g - \frac{\alpha\beta g s}{(\alpha + \beta)A},$$

for that of the pile, $\dfrac{d^2y}{dt^2} = g + \dfrac{\alpha\beta g s}{(\alpha+\beta)B} - \dfrac{Fg}{B}$,

subtracting, since $s = x - y$,

$$\frac{d^2s}{dt^2} = -\frac{\alpha\beta g s}{\alpha+\beta}\left(\frac{1}{A} - \frac{1}{B}\right) + \frac{Fg}{B} = -n^2 s + \frac{Fg}{B};$$

putting $n^2 = \dfrac{\alpha\beta g}{\alpha+\beta}\left(\dfrac{1}{A} + \dfrac{1}{B}\right) = \dfrac{\alpha\beta g}{\alpha+\beta}\,\dfrac{A+B}{AB}$.

Integrating, $s = \dfrac{Fg}{n^2 B} + C\cos nt + D\sin nt$,

when C, D are to be determined by the circumstances.

Suppose $t = 0$, when the pile begins to move. Then, as we have shewn, at this time $s' = \dfrac{\alpha+\beta}{\alpha\beta}(F - B)$: hence

$$\frac{\alpha+\beta}{\alpha\beta}(F-B) = \frac{Fg}{n^2B} + C = \frac{\alpha+\beta}{\alpha\beta}\,\frac{FA}{A+B} + C.$$

Therefore $C = \dfrac{\alpha+\beta}{\alpha\beta}\left(\dfrac{FB}{A+B} - B\right)$.

Again when $t = 0$, the velocity of the pile $= 0$, and that of the ram $= V'$.

Therefore at this time,

$$\frac{ds}{dt} = \frac{dx}{dt} - \frac{dy}{dt} = V' - 0 = V'.$$

But generally $\dfrac{ds}{dt} = -Cn\sin nt + Dn\cos nt$.

Hence making $t = 0$, $V' = Dn$, $D = \dfrac{V'}{n}$.

Hence, substituting for C, D, and n,

$$s = \frac{\alpha+\beta}{\alpha\beta}\,\frac{FB}{A+B} + \frac{\alpha+\beta}{\alpha\beta}\left(\frac{FB}{A+B} - B\right)\cos nt + \frac{V'}{n}\sin nt.$$

And substituting this in the value of $\frac{d^2y}{dt^2}$,

$$\frac{d^2y}{dt^2} = g\left(1 - \frac{F}{A+B}\right)(1 - \cos nt) + \frac{V'g\alpha\beta}{(\alpha+\beta)nB}\sin nt;$$

or, putting $g\left(\frac{F}{A+B} - 1\right) = m$,

$$\frac{d^2y}{dt^2} = -m + m\cos nt + \frac{V'g\alpha\beta}{(\alpha+\beta)nB}\sin nt.$$

Integrating twice, making $\frac{dy}{dt}$ and y both $= 0$ when $t = 0$, we have

$$\frac{dy}{dt} = -\frac{m}{n}(nt - \sin nt) + \frac{V'g\alpha\beta}{(\alpha+\beta)n^2B}1 - (\cos nt)$$

$$y = \frac{m}{n^2}(1 - \tfrac{1}{2}n^2t^2 - \cos nt)$$

$$+ \frac{V'g\alpha\beta}{(\alpha+\beta)n^3B}(nt - \sin nt)\ldots(2).$$

Now when the pile ceases to move, we have $\frac{dy}{dn} = 0$.

Hence

$$\frac{V'g\alpha\beta}{(\alpha+\beta)mn^4B}(1 - \cos nt) + \sin nt - nt = 0\ldots(3).$$

Whence t is determined: and t being known we have y, the whole space through which the pile is driven.

If we suppose V' to be small, that is, the velocity to be such as only just to stir the pile, we may approximate.

In this case t, the time during which the pile moves, will be very small. Hence expanding $\sin nt$ and $\cos nt$ in (3), and taking only the lowest two terms,

$$\frac{V'g\alpha\beta}{(\alpha+\beta)\,mnB}\,\frac{n^2t^2}{2} - \frac{n^3t^3}{2\,.\,3} = 0;$$

$$\text{therefore } t = \frac{3\,V'g\alpha\beta}{(\alpha+\beta)\,mn^2B};$$

or, putting for n^2 and m their values,

$$t = \frac{3\,V'A}{g(F-A-B)}$$

Hence find y from (2), again taking the lowest term only,

$$y = \frac{V'g\alpha\beta}{(\alpha+\beta)\,n^3B}\cdot\frac{n^3t^3}{2\,.\,3};$$

$$\text{or } y = \frac{gA^3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)B(F-A-B)}\cdot\frac{V'^4}{g^2}.$$

Cor. 1. If the ram will just stir the pile, V' in (1) is small, and a slight increase of the hardness of the ram or pile, (that is, of α or β,) or of the weight of the ram (that is, of A), will very much increase V', and thus increase very considerably the distance y, to which the pile is driven.

Cor. 2. The resistance F being supposed great, the space y will be very nearly inversely as to the weight of the pile; consequently the lighter the pile is, cæteris paribus, the faster it will be driven.

Cor. 3. The space y varies directly as the cube of the weight of the ram, the velocity with which the pile begins to move being given. And since this velocity itself is much increased by increasing the ram, there is on both accounts a great advantage in making the ram heavy.

211. Prop. *In the direct impact of inelastic bodies, vis viva is lost.*

Let A, B be the two bodies; u, v their velocities before the impact; w, the common velocity after the impact. Then

$$Au + Bv = (A + B)\,w,$$
$$A\,(u - w) = B\,(w - v).$$

Also $u + w - (u - w) = (w + v) + (w - v)$.

Multiplying

$$A\,(u^2 - w^2) - A\,(u - w)^2 = B\,(w^2 - v)^2 + B\,(w - v)^2.$$

Whence

$$Au^2 + Bv^2 - A\,(u - w)^2 - B\,(w - v)^2 = (A + B)\,w^2.$$

And $Au^2 + Bv^2$ is the vis viva before impact; and $(A + B)\,w^2$ is the vis viva after impact; therefore in the impact a quantity of vis viva is lost equal to

$$A\,(u - w)^2 + B\,(w - v)^2.$$

Cor. 1. The vis viva lost is that which corresponds to the sudden change of velocity.

Cor. 2. Since $w = \dfrac{Au + Bv}{A + B}$, $u - w = \dfrac{B\,(u - v)}{A + B}$,

$$w - v = \frac{A\,(u - v)}{A + B}.$$

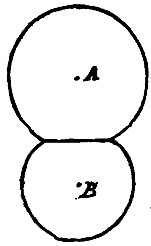
Hence the vis viva lost is

$$\frac{AB^2 + BA^2}{(A + B)^2}\,(u - v)^2 = \frac{AB}{A + B}\,(u - v)^2,$$

the quantity $\dfrac{AB}{A + B}$ is called a *harmonic mean* between A and B. And it appears that *the vis viva lost in collision, is that corresponding to a body equal to the harmonic mean of the bodies moving with the difference of the velocities.*

212. Prop. *In the direct collision of two inelastic or imperfectly elastic bodies, the quantity of vis viva lost is double of the total labouring force expended in the compression.*

Let A, B be the centers of the bodies not affected by the compression; let a, b, be the distances which would intervene between those centers and the point of contact without compression; $a - p$, $b - q$, the distances at any period of the contact. Also let x and y be the distances of A and B from a fixt point in the direction of the motion and of the impact. Then



$$y - x = a + b - p - q$$

$$\frac{dy}{dt} - \frac{dx}{dt} = -\left(\frac{dp}{dt} + \frac{dq}{dt}\right);$$

or if u, v be the velocities of A, B,

$$u - v = -\left(\frac{dp}{dt} + \frac{dq}{dt}\right).$$

Let P be the mutual pressure of the bodies: the effect of P on the motion of the centers A, B, is the same as if it were applied directly to each. Hence

$$\frac{du}{dt} = -\frac{Pg}{A}, \quad \frac{dv}{dt} = \frac{Pg}{B}.$$

Whence

$$2Au\frac{du}{dt} + 2Bv\frac{dv}{dt} = 2Pg(v - u) = 2Pg\left(\frac{dp}{dt} + \frac{dq}{dt}\right),$$

$$\text{and } Au^2 + Bv^2 = C - 2g\int_t P\left(\frac{dp}{dt} + \frac{dq}{dt}\right).$$

If U, V be the velocities before impact, and if the integral begin with the impact,

$$Au^2 + Bv^2 = AU^2 + BV^2 - 2g \int_t P \left(\frac{dp}{dt} + \frac{dq}{dt}\right).$$

Now the integral $\int_t P\frac{dp}{dt}$ is the labouring force which is consumed when a pressure P acts through a space p; that is, the force consumed is the compression of A. In like manner $\int_t P\frac{dq}{dt}$ is the force consumed in the compression of B. Hence the Proposition is manifest.

213. Prop. *Knowing the law of compression in the impact of inelastic bodies: to find its amount.*

As before, let the pressure due to the compressions p, q, be αp, βq. Therefore

$$2\int P\frac{dp}{dt} = \alpha p^2, \quad 2\int P\frac{dq}{dt} = \beta q^2.$$

$$Au^2 + Bv^2 = AU^2 + BV^2 - g(\alpha p^2 + \beta q^2).$$

But since the bodies are inelastic,

$$u = v = \frac{AU + BV}{A + B}.$$

Also as before,

$$\alpha p = \beta q, \text{ whence } q = \frac{\alpha p}{\beta}.$$

Hence we find, $g\dfrac{(\alpha + \beta)}{\beta} p^2 = \dfrac{AB}{A + B}(U - V)^2.$

Hence the masses and velocities being known, p is known.

Cor. 1. Let $A = B$ and $\alpha = \beta$,

$$4\alpha g p^2 = A(U - V)^2.$$

Cor. 2. In this case let c^2 be the section of the body A, perpendicular to the direction of impact, and mc its length in that direction; therefore, by Art. 207,

$$a = \frac{Ec}{m}; \ 4Ecgp^2 = mA(U-V)^2;$$

$$p = (U-V)\sqrt{\frac{mA}{4Ecg}}.$$

Ex. Let the bodies be iron cubes.

Then $m = 1$, $E = 9{,}000000$; $g = 32$, $A = c^3$; therefore

$$p = (U-V)\frac{c}{24000\sqrt{2}}.$$

Let the difference of velocity be 1000 feet per second; and the side of the cubes 1 foot;

$$p = \frac{1}{24\sqrt{2}} \text{ of a foot} = \frac{1}{3} \text{ of an inch, nearly.}$$

In this case the greatest pressure is ap = a column of iron of the height $\frac{9{,}000000}{24\sqrt{2}}$ feet = 265 feet, the base being 1 foot.

For bodies with convex surfaces, E will vary with p.

Cor. 3. In the case of inelastic bodies we have

$$2\int_t \left(P\frac{dp}{dt} + P\frac{dq}{dt}\right) = A(U-u)^2 + B(V-v)^2$$

at every period of the contact.

And when u, v, are both $= w$,

$$2\int_t \left(P\frac{dp}{dt} + P\frac{dq}{dt}\right) = A(U-w)^2 + B(V-w)^2.$$

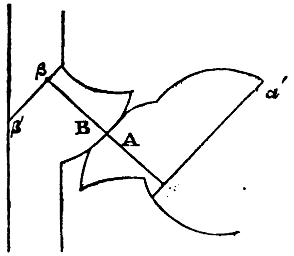
The force consumed in the compression of the two bodies is half the vis viva due to the sudden change of velocities.

214. Prop. *In any collision of inelastic parts of a system, a quantity of vis viva is lost, equal to the double of the labouring force expended in the compression of the parts.*

Let P, Q, be the parts between which collision takes place, AB perpendicular to the plane of contact, $\alpha\alpha'$, $\beta\beta'$,

two planes taken in the colliding parts, perpendicular to AB, beyond the range of the particles displaced by the collision.

Let co-ordinates x, y, be taken; x being parallel to AB, y perpendicular to it: and let a, b, be the distances of the plane of contact from the planes $\alpha\alpha'$, $\beta\beta'$, if there were no compression; p, q the compression of the body P, and Q, respectively. Hence x, x' being the values of x for the planes $\alpha\alpha'$, $\beta\beta'$,



$$\text{hence } x' - x = a + b - p - q;$$

$$\frac{dx'}{dt} - \frac{dx}{dt} = -\left(\frac{dp}{dt} + \frac{dq}{dt}\right).$$

Let a pressure P act at each point of the plane of contact; and let S indicate a sum taken for all these points. Therefore $S.P$ is the total pressure which acts upon the bodies A, B, in opposite directions.

Let m &c. be the particles of the system moving at any time t, with velocities $\frac{dx}{dt}$ &c. in the direction of x, and with velocities $\frac{dy}{dt}$ &c. in the direction of y: and let Σ represent the sum taken for all these particles. Hence the *effective moving forces* in the direction of x are $m\frac{d^2x}{dt^2}$ &c., and the sum of them all is $\Sigma.m\frac{d^2x}{dt^2}$.

Also the *impressed moving forces* in the direction of x, are $S.P$ upon one part, and $-S.P$ upon the other in contact with it. And for the virtual velocities, we may take the actual velocities. Hence, by the equilibrium of impressed and effective forces, we have

$$\Sigma . m \frac{dx}{dt} \frac{d^2x}{dt^2} = S . P \frac{dx'}{dt} - S . P \frac{dx}{dt}$$

$$= S . P \left(\frac{dx'}{dt} - \frac{dx}{dt}\right)$$

$$= - S . P \left(\frac{dp}{dt} + \frac{dq}{dt}\right),$$

Hence integrating, which we may do under both Σ and S, since they are merely sums of terms,

$$\Sigma . m \left(\frac{dx}{dt}\right)^2 = C - 2S \int_t \left(P \frac{dp}{dt} + P \frac{dq}{dt}\right).$$

Also there is no impressed force in the direction y.

$$\therefore \Sigma . m \frac{dy}{dt} \frac{d^2y}{dt^2} = 0; \ \Sigma . m \left(\frac{dy}{dt}\right)^2 = \text{constant}.$$

Hence

$$\Sigma . m \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = C - 2S \int_t \left(P \frac{dp}{dt} + P \frac{dp}{dt}\right).$$

Or if V be the velocity at first, and v after any time,

$$\Sigma . m v^2 = \Sigma . m V^2 - 2S \int_t \left(P \frac{dp}{dt} + P \frac{dq}{dt}\right).$$

If we were to consider the motion with regard to a third co-ordinate z, perpendicular to x and to y, we should still have the same result.

Hence the Proposition is manifest.

We have supposed the system acted on by the collision only. But if there be any other impressed forces, the result will be the same. The change of the vis viva, due to the impressed forces, will take place in addition to that due to the collision. For in this case the terms expressing the impressed forces must be added to the term involving P.

215. Prop. *In the collision of inelastic parts of a system, the quantity of vis viva lost is equal to that due to the sudden change of velocity of the colliding parts.*

If u be the velocity of a particle m in the direction x, v its velocity in the direction y, at any time; and if δx, δy, be the virtual velocities of the particle in these directions; then by the principle of virtual velocities, and the equilibrium of impressed and effective moving forces,

$$\Sigma . m \left(\frac{du}{dt}\delta x + \frac{dv}{dt}\delta y\right) = S\,(P\delta p + P\delta q).$$

The relation of δx, δy, varies with the motion of the system. But during the short time which the collision occupies, δx, δy may be conceived as constant. Hence integrating with regard to t,

$$\Sigma m \left\{(u - U)\,\delta x + (v - V)\,\delta y\right\} = S\,(\textstyle\int_t P\delta p + \int_t P\delta q),$$

where U, V are the values of u, v, when the collision begins.

But $\int_t P\delta p + \int_t P\delta q$ for the whole collision $= 0$. For δp, δq are the virtual velocities of the disturbed particles relative to the undisturbed, estimated in the direction of P. Now for these virtual velocities we may take the actual velocities $\frac{dp}{dt}$ and $\frac{dq}{dt}$, of which one is contrary to P's direction; therefore the sum of all the terms taken in P's direction is

$$S . P\left(\frac{dp}{dt} - \frac{dq}{dt}\right);$$

which is 0 at every instant, because the forces P acting on the one part and on the other are always equal.

Hence the integral of this with regard to t is 0; and we have

$$\Sigma . m \left\{(u - U)\,\delta x + (v - V)\,\delta y\right\} = 0.$$

But when the collision ends, and the parts move on together, u, and v, the actual velocities of m, may be put for its virtual velocities. Therefore

$$\Sigma . m \{(u - U) u + (v - V) v\} = 0,$$

$$\text{whence } \Sigma . m (u^2 + v^2 - Uu - Vv) = 0.$$

Hence the vis viva lost

$$= \Sigma . m (U^2 + V^2 - u^2 - v^2),$$

$$= \Sigma . m \{(U - u)^2 + (V - v)^2 - 2 (u^2 + v^2 - Uu - Vv)\},$$

$$= \Sigma . m \{(U - u)^2 + (V - v)^2\}, \text{ by what has been proved.}$$

And since $(U - u)^2$ is the square of the velocity lost in the direction of x, and $(V - v)^2$ the square of the velocity lost in the direction of y, $(U - u)^2 + (V - v)^2$ is the square of the velocity lost, obtained by resolving the velocities according to rectangular co-ordinates.

Whence the Proposition is manifest.

216. Since in all collision among the parts of a moving system labouring force is lost, all such collision is a disadvantage in the working of a machine, and is to be avoided as much as possible. The preceding principles enable us to calculate the loss of labouring force incurred by collision, and the gain resulting from any contrivance which removes collision. This will appear in the following examples.

(1). Let a weight be dragged along a rough road, and a smooth one; to find the difference of labouring force consumed to produce the same velocity.

We suppose the weight to move preserving its parallelism. Let the roughness of the road be expressed by supposing it to consist of small obstacles of the height h, and of which there are n in each unit of length. Let w be the weight, v its velocity, f the coefficient of friction: then fwv is the labouring force expended in a unit of length on

the smooth road: but in going over each of the obstacles there is expended a vis viva $w \times 2gh$, and therefore a labouring force wgh, and in a unit of space the labouring force expended is $nwgh$. Hence the labouring force on the smooth and on the rough road are in the proportion

$$fwv : fwv + nwgh, \text{ or } 1 : 1 + \frac{ngh}{fv}.$$

(2). If in addition to the labouring force required in order to rise over the obstacle, there be horizontal velocity lost by impinging against the obstacle, let the velocity lost at each obstacle be a fraction m of the velocity of impact. Hence the vis viva lost in each impact is wv^2m^2; and in a unit of space it is wv^2m^2n: which must be supplied in order to keep up the velocity v. Hence the labouring force required on this account is $\frac{1}{2} m^2nwv^2$.

(3). If the body dragged, instead of moving parallel to itself, have parts which *turn* over the obstacle, as the wheels of a carriage; the different portions of those parts will rise through different heights; and the whole labouring force lost will be $\Sigma . gwh$; each part w being multiplied into the height h through which it rises.

(4). If the surface on which the body is dragged be made a curve, no sudden turns or finite angles between two successive elements occurring, there will be no horizontal velocity lost; and the labouring force which was found to be requisite on this account, will not be needed.

This may be attained by giving to all the obstacles a curved outline sloping at a small angle: so that at the rise, the concavity shall be upwards, and the curvature less than that of the under part of the body or its wheels.

(5). This is often not possible. But the same effect is attained by springs, for the part of the weight supported

on springs describes a curve, and not a line with sudden turns. Hence the gain by the use of springs is $\frac{1}{2}m^2 n w v^2$.

(6). If a wheel of radius r come in contact with an abrupt obstacle of which the height is h, the point of contact will be distant from the lowest point by an angle a, of which the versed sine is h to radius r. Also at the first instant after contact, the center of the wheel moves in a direction making an angle a with the horizon; and therefore the horizontal velocity retained will be $v \cos a$, or $v\left(1-\frac{h}{r}\right)$. Hence in the above expression, $m=\frac{h}{r}$; and the labouring force required to travel over n such obstacles is $\frac{nwv^2h^2}{2r^2}$.

(7). If we suppose the obstacles to be paving stones of a given shape, their length will be proportional to their height; nh is constant, and the labouring force required in a unit of space is as $\frac{wv^2h}{r^2}$, or as $\frac{wv^2}{nr^2}$.

Therefore it is as the square of the velocity and the height of each stone directly, and as the square of the radius of the wheel inversely.

217. The force which is thus lost is consumed, as we have proved, in compressing the bodies which impinge. Its effects are not obvious, for how many strokes does it require to make a hammer shew any visible signs of compression! But the smallness of the effect arises from the largeness of the modulus of elasticity. That there is really such an effect may be readily seen; for a hammer, after being long used, has its surface and its substance modified. And softer substances, as wood, shew more readily the effects of impact in *disfiguring* them. Also roads and carriages, by long usage, are damaged and destroyed.

218. When the bodies which impinge are imperfectly elastic, a part of the effect of the compressing force is reproduced in the velocity with which they separate. But when bodies are inelastic, the whole of the compressing force disappears, being absorbed by the internal constitution of the body, and can never be brought out in the form of labouring force.

219. The loss of force which takes place in impact, makes it desirable to construct machines so that they shall as much as possible operate without collision of parts. Thus if a wheel armed with cams lift a hammer, if the cam impinge against the hammer at every stroke, force is lost; but if the cam and the corresponding tooth of the hammer can be made of such a form that the cam shall gradually bring the hammer into motion, rolling along the tooth, the loss of force by impact is avoided.

220. The same principles apply to fluids as well as solids. Hence when water is used as a moving force and impinges so that it has to change its velocity suddenly, a portion of the force is lost, being consumed in the interior commotion of the water. But if the water gradually acquires the velocity of the wheel on which it acts, the whole of its labouring force is brought into play.